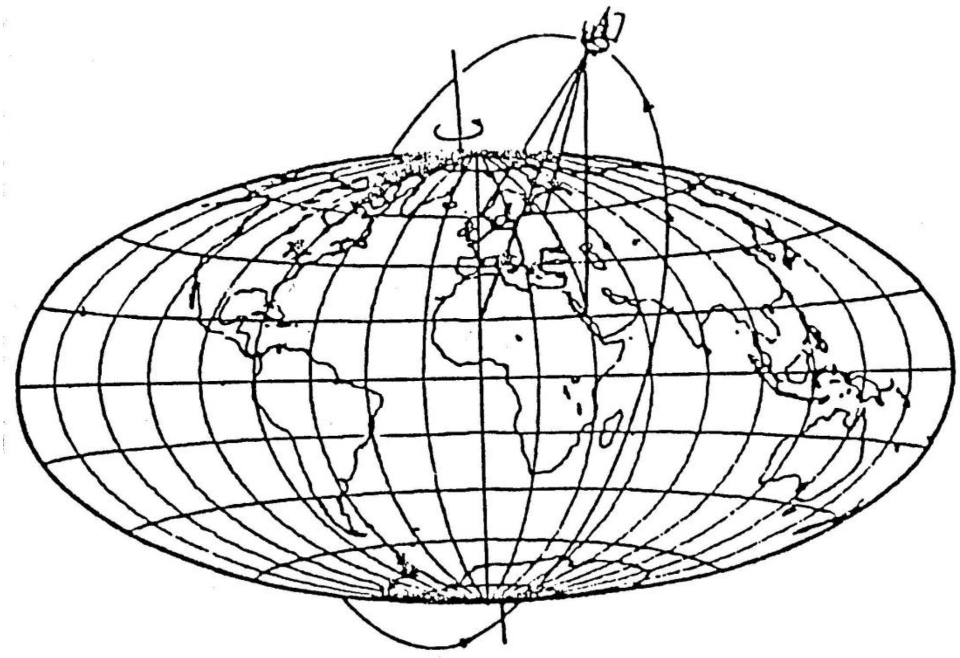

# G E O D E S Y

## Part 2


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# Chapter 5

## Reference systems

### 5-1 Introduction

It is known that by means of certain mathematical operations we can transfer our physical observations into geodetic information of position, azimuth, elevation, distance, or size and shape of the earth.

A coordinate system is necessary for all our calculations , whether as an intermediate step or end result. Such a reference system may take many forms of which sometimes one, and sometimes another may be the most convenient. Some of these systems, which are of special importance in geodesy, may be described briefly as follows:

### 5-2 Natural Coordinate System $\Phi, \Lambda, H$

Astronomical observations for latitude, longitude, and azimuths are measured with reference to the direction of gravity at the point of observation. In the natural coordinate system the position of any point on the earth's surface can be fixed by observing its astronomic latitude, longitude, and its orthometric height, figure (5-1).

1) Astronomical Latitude $\Phi$: is the angle between the equatorial plane and the direction of the vertical at the point of observation.

2) Astronomical Longitude $\Lambda$: is the angle between the meridian plane of the observation point and the meridian plane of Greenwich.

3) Orthometric Height $H$: is the height of a point above mean sea level. It is measured along the curved plumb line and obtained from spirit levelling and gravity observations.

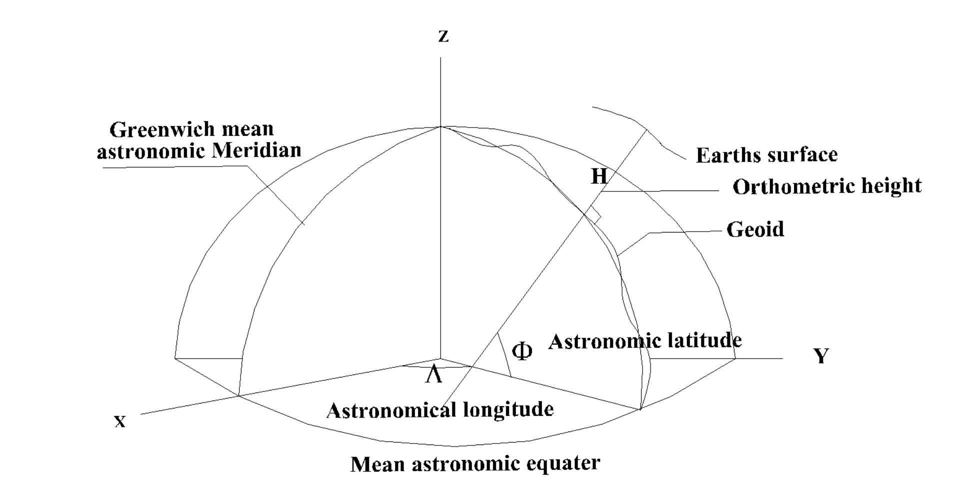



FIG 5-1

Consequently, the quantities $\Phi$, $\Lambda$, and $H$ define the position of the observer with respect to the geoid & the mean rotational axis of the earth.

## 5-3 Geodetic Coordination System $\phi, \lambda, h$

Since the deviations of the geoid from the reference ellipsoid are small and can be computed, it is convenient to add small reductions to the observed coordinate so that, values refer to an ellipsoid can be established, which are called geodetic coordinates, figure (5-2).

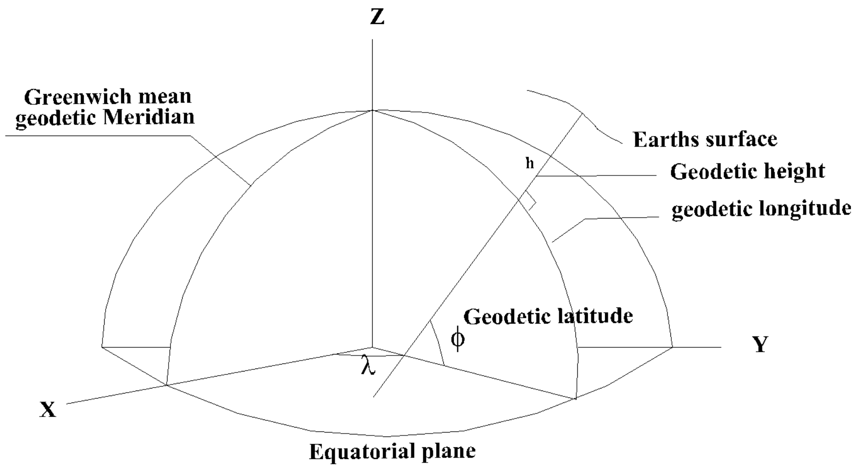



FIG 5-2

1) Geodetic Latitude $\phi$: is the angle between the ellipsoidal normal of the observers projected position on the geoid and the perpendicular to the mean rotation axis of the earth.

2) Geodetic Longitude $\lambda$: is the angle between the same ellipsoidal normal and Greenwich meridian plane.

3) Geodetic Height $h$: is the height of the observer above the reference ellipsoid, measured along the ellipsoidal normal.

The geodetic coordinates are determined from *Triangulation* or *Trilateration* observed on the earth surface, reduced to the ellipsoid.

They could also be obtained directly from the astronomic coordinates reduced to the used reference ellipsoid.

## 5- 4 Rectangular Coordinate System X, Y, Z.

Generally, it is convenient to take the X-axis parallel to the meridian of Greenwich; the Y-axis has the longitude of 90˚ east of Greenwich, and the Z-axis parallel to the CIO (conventional international origin of polar motion). Ideally the origin of the rectangular coordinates system should be at the earth's center of gravity; the system is known as "Average Terrestrial Coordinate System". When the origin is at the geometric center of the ellipsoid, and not in the (C.G.) of the earrth, it is known as "Geodetic Coordinate System" figure (5-3).

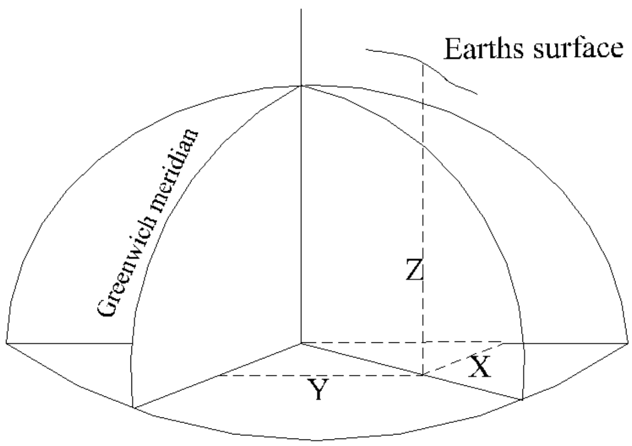



FIG 5-3

## 5-5 Local Coordinate System (Horizon System) U, V, W.

In this system the coordinates U, V, W are expressed as functions of the observed azimuth A, zenith distances Z & spatial distance S. figure (5-4)

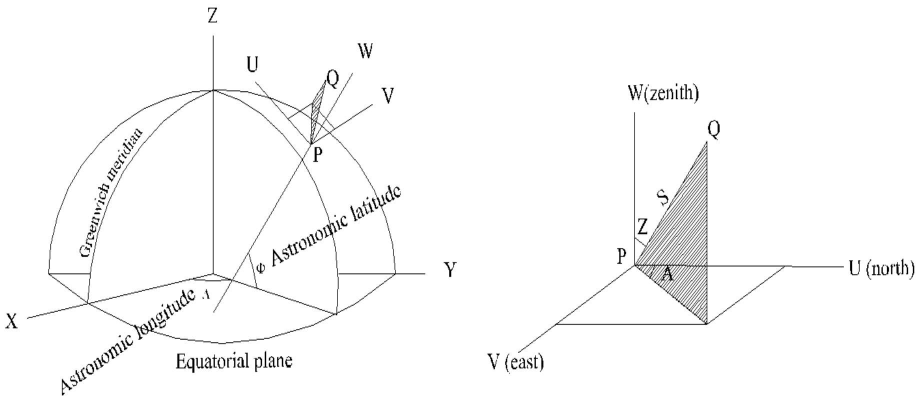



FIG 5-4

illustrates the quantities of this system. The origin is considered to be at the observation station P. the positive U-axis points N-ward, the positive V-axis points eastward and the positive W-axis coincides with the outward direction of the plumb line.

The coordinate equations of an object Q sighted in this system may be written simply by referring to the fig.

$$U = S \sin Z \cos A$$

$$V = S \sin Z \sin A \qquad (5\text{-}1)$$

$$W = S \cos Z$$

## 5- 6 Relations between different Reference Systems

There are certain mathematical relations connecting the previous coordinate systems. These relations are among the basic equations in geodesy. Some of these relations will be considered in the next subsections.

## 5-6-1 Relation between Astronomic and Geodetic Coordinates

Since the astronomical system depends on the direction of the vertical "actual gravity field", while the geodetic system depends on the direction of the ellipsoidal normal "normal gravity field", then the relation between both systems depends mainly on the difference between the two directions. The total difference between the two directions is the well-known deflection of the vertical $\theta$. It has two components, a north-south component $\xi$ and an east-west component $\eta$. We can read from figure (5-5) the following:

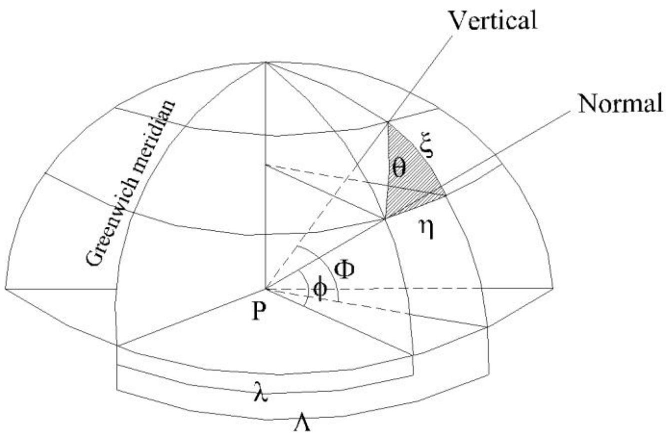



FIG 5-5

$$\xi = \Phi - \phi \tag{5-2a}$$

$$\eta = (\Lambda - \lambda)\cos\phi \tag{5-2b}$$

$$\theta = (\xi^2 + \eta^2)^{0.5} \tag{5-2c}$$

According to Helmert's projection, which neglects the curvature of the plumb line, a point $P$ on the earth's surface is directly projected onto the ellipsoid by means of straight ellipsoidal normal, point $P_1$. Then the ellipsoidal height is given by

$$h = H + N \tag{5-3}$$

In pursuing this relationship, it is important to remember Pizzetti's projection, figure (5-6). In this projection the same point P is projected along the curved plumb line onto

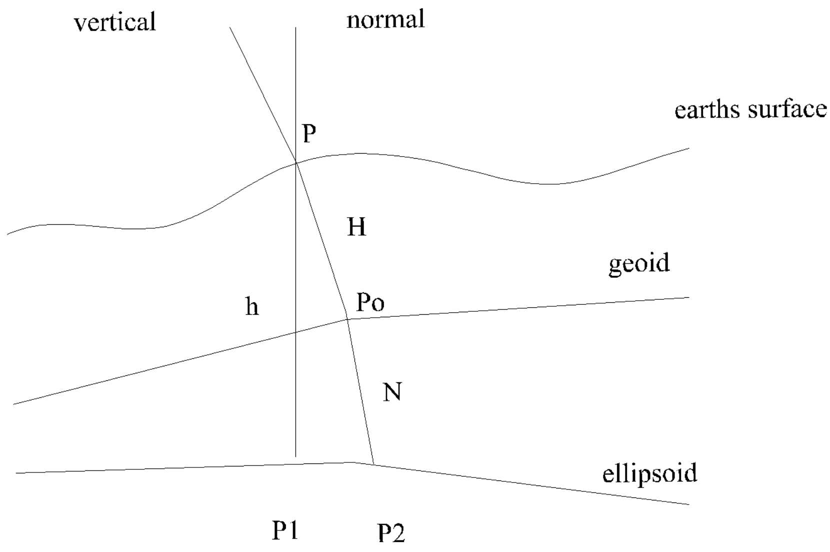


FIG 5-6

the geoid, point Po, and then projected onto the ellipsoid, point P2. The practical difference between the two projections is small, and within a fraction of millimeters.

## 5-6-2 Relation between Rectangular and Curvilinear Geodetic Coordinates:

The coordinate transformation between the curvilinear geodetic coordinates and the Cartesian coordinates may be expressed symbolically by

$$(\phi,\lambda,h)\xleftrightarrow[(a,f)]{}(X,Y,Z)$$

From figure (5-7) the relation between the two systems can be written as follows;

$$R_P = (N+h)\cos\phi \qquad (5\text{-}4)$$

$$X = R_P \cos\lambda$$

$$Y = R_P \sin\lambda$$

where,

$N =$ Radius of curvature in the prime vertical

$$N = \frac{a}{(1-e^2\sin^2\phi)^{0.5}}$$

$h =$ Ellipsoidal height

Also, from same figure, we can read

$$Z = Z_1 - K$$

where,

$Z_1 = (N + h)\sin\phi$

$K = N \cdot e^2 \sin\phi$

Combining the latter expression with the formers, we get:

$$X = (N + h)\cos\phi\cos\lambda \qquad (5\text{-}5a)$$

$$Y = (N + h)\cos\phi\sin\lambda \qquad (5\text{-}5b)$$

$$Z = (N(1 - e^2) + h)\sin\phi \qquad (5\text{-}5c)$$

These equations are the basic transformation formulas between the geodetic coordinates $\phi, \lambda, h$ and the rectangular coordinates $X, Y, Z$ of a point outside the ellipsoid. The origin of the rectangular coordinate system is the center of the ellipsoid, and the Z-axis is its axis of rotation; the X-axis has the Greenwich $0^\circ$ longitude and the Y-axis has longitude $90^\circ$ east of Greenwich (i.e., $\lambda = +90^\circ$).

## Inverse Procedure:

The computation of $\phi, \lambda, h$ from given $X, Y, Z$ is more complicated because the three equations have four unknowns, N including $\phi$. Accordingly, the computation could be done iteratively in addition to the direct solution. Many solutions, through iteration, were given for this problem, for example; *HIRVONEN & MORITZ 1963 BARTELME & MEISEL 1975, RAPP* and *KRAUSS 1976*. Also a non-iterative solution was given by *SUENKEL 1976*.

For the iterative solution we shall follow *(Hirvonen & Moritz 1963)*. Now from figure (5-7) we find

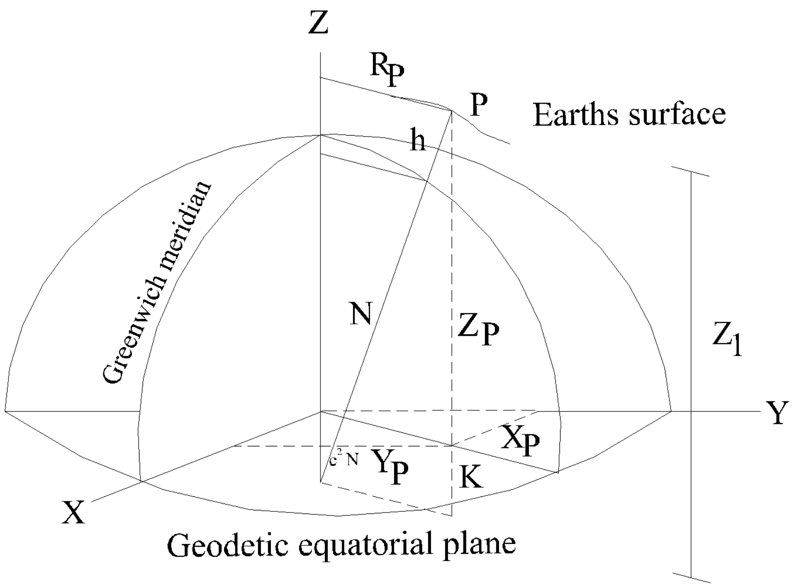



FIG 5-7

$$R_P = (X^2 + Y^2)^{0.5} = (N + h)\cos\phi$$

hence

$$h = \frac{R_P}{\cos\phi} N \qquad (5\text{-}6)$$

Equation (5-5) may be transformed into

$$Z = (N - \frac{a^2 - b^2}{a^2} N + h)\sin\phi$$

$$Z = (N + h - e^2 N)\sin\phi$$

where

$$e^2 = (a^2 - b^2) / a^2$$

Dividing this equation by the above expression for $R_P$ we get

$$\frac{Z}{R_P} = (1 - e^2 \frac{N}{N+h})\tan\phi$$

so that

$$\tan\phi = \frac{Z}{R_P}(1 - e^2 \frac{N}{N+h})^{-1} \qquad (5\text{-}7)$$

Given $X, Y, Z$ and hence, $R_P$ equations (5-6) and (5-7) may be solved iteratively for $h$ and $\phi$.

As a first approximation, we set $h = 0$ in (5-7), obtaining

Using $\phi_1$, we compute an approximate value $N_1$ by means of

$$N_1 = \frac{a}{(1 - e^2 \sin^2 \phi_1)^{0.5}} \qquad (5\text{-}8)$$

and introduce this value of $N_1$ in equation (5-6) to get an approximate value $h_1$.

$$h_1 = \frac{R_P}{\cos\phi_1} - N_1$$

Now, as a second approximation, we set $h = h_1$ in (5-7) obtaining

$$\tan\phi_2 = \frac{Z}{R_P}(1-e^2\frac{N_1}{N_1+h_1})^{-1}$$

Using $\phi_2$, improved values for N & h are found, etc. This procedure is repeated until the values of $\phi$ & h remain practically constant. The third value $\lambda$ can be easily calculated from

$$\tan\lambda = Y/X$$

$$\tan\phi_1 = \frac{Z}{R_P}(1-e^2)^{-1} \qquad (5\text{-}9)$$

## 5-6-3 Relation between Horizon and Rectangular System

Since all the observations in geodesy, mainly horizontal, vertical angles, and spatial distances, are made with respect to the direction of the vertical at the observation station. Then it is important to find out the relations connecting these observable quantities of these two systems. Figure (5-8) illustrates the quantities of these two systems, where point $P$ represents the occupied station, $Q$ is the observed objects, and ***pq*** is the horizontal projection of the spatial distance $S$ onto the horizon plane ***Psqn*** of the local system $U,V,W$. Then by equation (5-1) we can compute $u,v,w$ of any point $Q$ from station $P$. The plane through points $M,O,q'$ is parallel to the equatorial plane of the $X,Y,Z$ system. The projection of this horizon plane on $MOq'R$ plane is given as follows

$$MO = NO + MN$$

$$MO = w\cos\Phi - u\sin\Phi \qquad (5\text{-}10)$$

Likewise

$$\begin{aligned}\Delta X &= Ma - Ob = MO\cos\Lambda - v\sin\Lambda \\ \Delta Y &= aO - bq = MO\sin\Lambda + v\cos\Lambda \\ \Delta Z &= -PM + qQ = u\cos\Phi + w\sin\Phi\end{aligned} \qquad (5\text{-}11)$$

Then, the final form are achieved by combining these relations together as follows

$$\begin{vmatrix}\Delta X\\ \Delta Y\\ \Delta Z\end{vmatrix} = \begin{vmatrix}-\sin\Phi\cos\Lambda & -\sin\Lambda & \cos\Phi\cos\Lambda\\ -\sin\Phi\sin\Lambda & \cos\Lambda & \cos\Phi\sin\Lambda\\ \cos\Phi & 0 & \sin\Phi\end{vmatrix} \cdot \begin{vmatrix}u\\ v\\ w\end{vmatrix}$$

Or in matrix notation $\Rightarrow X = R^T \cdot u$

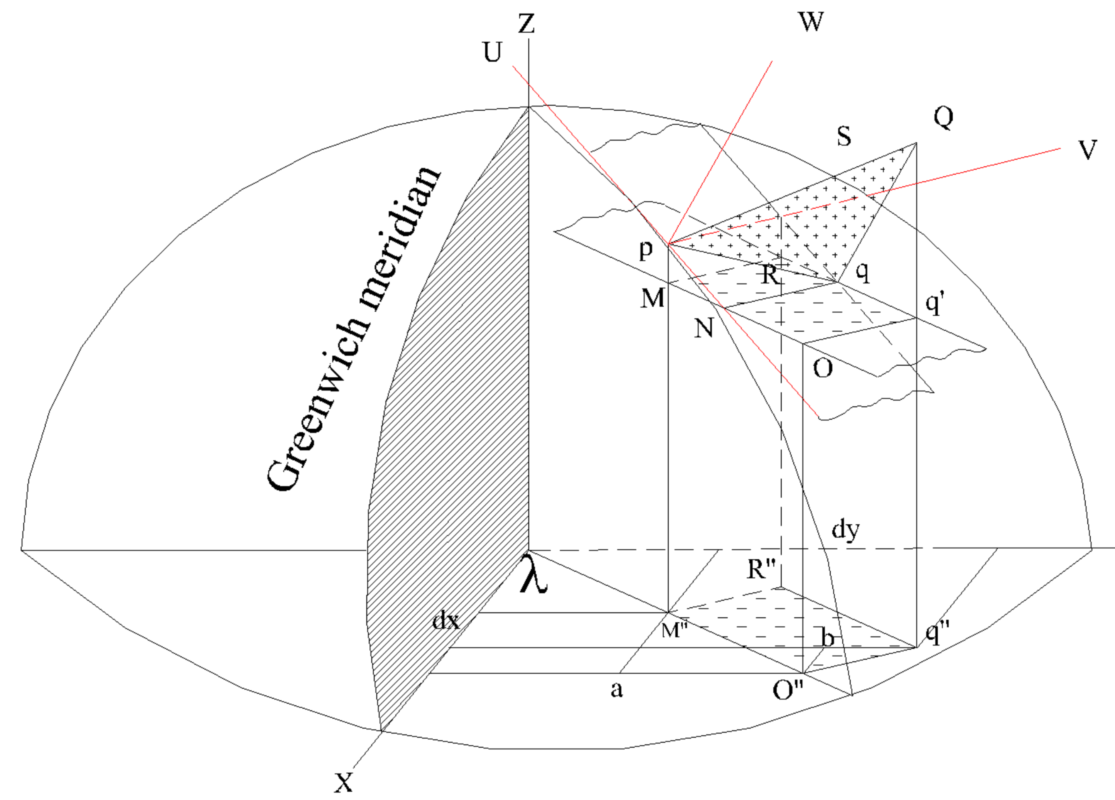



FIG 5-8

# Chapter 6

## Effect of the direction of the gravity vector on the geodetic computations "gravimetric effect"

We all know the principle of triangulation, where distances and elevations are obtained indirectly by measuring the horizontal and vertical angles in a suitable network of triangles. Only one base line is necessary to furnish the scale of the network, and one azimuth furnishes the orientation of it. The computation of triangulations on the ellipsoid is easy. But, we must not forget that our observation were taken relative to the direction of the gravity vector, while the computations will be carried out on the surface of the ellipsoid using fictitious observations taken relative to the direction of the normal to the ellipsoid, it is therefore convenient to reduce the measured angles, base lines, and long distances to the ellipsoid, in much the same way as the astronomical coordinate are treated.

The new observations after reduction can then be used to calculate the geodetic coordinates $\phi, \lambda, h$ for each point of the triangulation net. One should not forget that during the calculation the geometrical effect of the ellipsoid on the observations should also be taken into consideration.

### 6-1 Gravimetric Effect on Astronomic Azimuth

Let us consider a celestial sphere, figure (6-1), with its center at the observation station $P$, and the actual plumb line intersects this sphere at the astronomical zenith $Z_a$, whereas the ellipsoidal normal intersects it at the geodetic zenith $Z_g$. The line of sight to the target for which the azimuth $A$ is measured, intersects this sphere at point $T$ and has the zenith distances $Z'$ and $Z$ with respect to $Z_a$ and $Z_g$. $P_N$ Corresponds to the direction of the North Pole, which has the zenith distances $90-\Phi$ and $90-\phi$ with respect to $Z_a$ and $Z_g$.

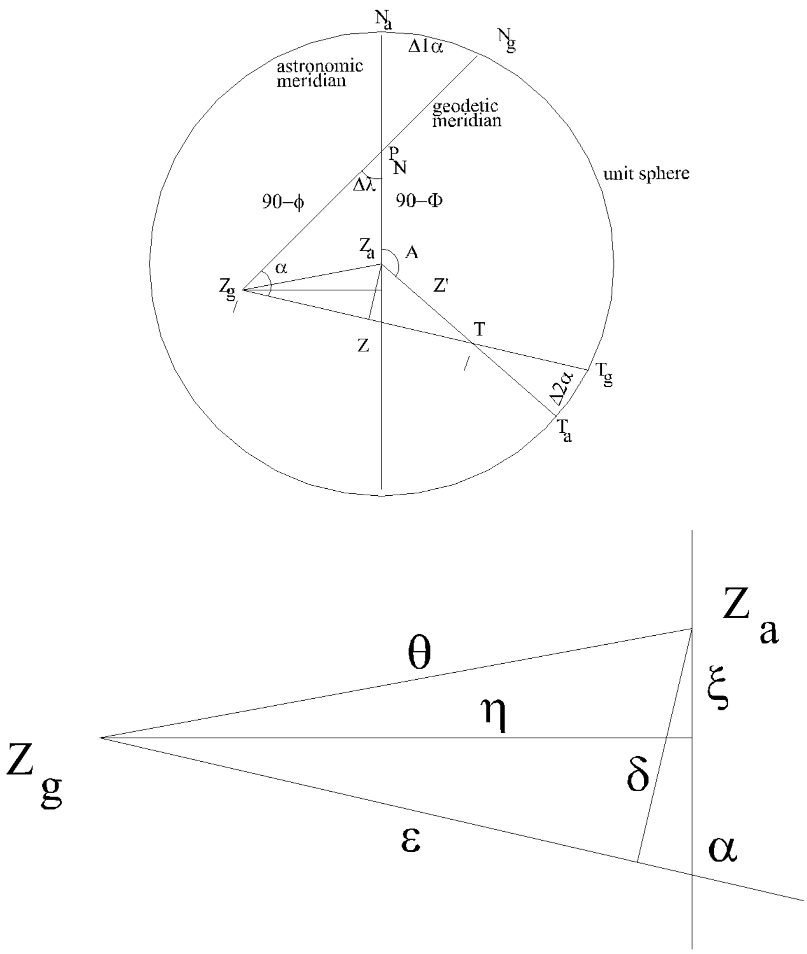



FIG 6-1

From figure (6-1), the angle at $P_N$ is the difference between the astronomical and the geodetic longitude.

$$\Delta\lambda = \Lambda - \lambda \tag{6-1}$$

The angle at $Z_a$ ($P_N\ Z_a\ T$) is the astronomical azimuth $A$ for target $T$, and the angle at $Z_g$ ($P_N\ Z_g\ T$) is the geodetic azimuth $\alpha$ for the same target $T$.

The point $F$ lies on the astronomical meridian, the great circle connecting $P_N$ and $Z_a$, so that the angle $Z_aFZ_g$ is $90^\circ$. The components of the deflection of the vertical are as follows

$\xi = Z_aF$ & $\eta = Z_gF$

The difference between astronomic and geodetic azimuth is given by

$$\Delta\alpha = A - \alpha \tag{6-2}$$

and it consists of two parts $\Delta_1\alpha$, and $\Delta_2\alpha$ fig. (2-1)

The value of $\Delta_1\alpha$ is obtained from the spherical triangle $N_g\ N_a\ P_N$ by Napier's rule as follows:

$$\sin\Delta_1\alpha = \cos(90-\Delta\lambda)\cos(90-\phi) \tag{6-3}$$

For the small angles we can use the approximation

$\sin\Delta_1\alpha = \Delta_1\alpha$

$\cos(90-\Delta\lambda) = \Delta\lambda$

Thus we find

$$\Delta_1\alpha = \Delta\lambda\sin\phi \tag{6-4}$$

Or using equation (6-2b) together with (6-4) we get

$$\Delta_1\alpha = \eta\tan\phi \tag{6-5}$$

On introducing a point $G$ on the great circle connecting $Z_g$ and $T$ so that the angle and ($Z_aGZ_g$) is $90$, putting ($Z_aG$) equals $\delta$, we see that the figure ($Z_aGTT_gT_a$) has the same geometry as the figure ($Z_gFPN_aN_g$), so that $\Delta_2\alpha$, $\delta$, $Z'$ correspond to $\Delta_1\alpha, \eta, 90-\phi$.

Accordingly the equation corresponding to (6-5) is thus

$$\Delta_2\alpha = \delta\cot Z' = \delta\cot Z \tag{6-6}$$

Since the small figure ($Z_aFZ_gG$) may be considered plane we get by the usual formula of transformation of plane coordinates:

$$\delta = \xi\sin\alpha - \eta\cos\alpha \tag{6-7}$$

So that

$$\Delta_2\alpha = (\xi\sin\alpha - \eta\cos\alpha)\cot Z \tag{6-8}$$

Now combining (6-5) & (6-8) we obtain

$$\Delta\alpha = \eta\tan\phi + (\xi\sin\alpha - \eta\cos\alpha)\cot Z \tag{6-9}$$

Equation (6-9) is the well-known "Laplace Equation" in its complete form; the first term is the same for every target independent of its azimuth and zenith distance. And it results from the fact that astronomical north $N_a$ rather than from geodetic north $N_g$ as the geodetic azimuth. It thus represents a shift of the zero point, which is the same for all targets. The second term arises because the target $T$ is projected from $Z_a$ and $Z_g$ onto different points $T_a$ and $T_g$ of the horizon; the effect is the same as that of an inaccurate levelling of the theodolite.

Usually in first-order triangulation the lines of sight are almost horizontal, so that $Z = 90^\circ, \cot Z = 0$. Therefore, the correction can in general be neglected & we thus get

$$\Delta\alpha = \eta\tan\phi = \Delta\lambda\sin\phi \tag{6-10}$$

This is also the Laplace equation but in its simplified form. It is remarkable that the differences

$\Delta\alpha = A - \alpha$ and $\Delta\lambda = \Lambda - \lambda$

Should be related in such a simple way

Recall now the enlarged section of figure (6-1), this section will be used to illustrate the way of computing the deflection of the vertical components in any direction $\alpha$, $\delta$ and $\varepsilon$ are related to $\zeta$ and $\eta$ by plane coordinate transformation as follows

$$\varepsilon = \xi \cos\alpha + \eta \sin\alpha \tag{6-11}$$

$$\delta = \xi \sin\alpha - \eta \cos\alpha \tag{6-12}$$

## 6-2 Gravimetric Effect on Vertical Angles

As we have seen from the previous discussion that the component of the deflection of the vertical in any direction is given by eq. (6-11).

Figure (6-2) is another illustration for the astronomic "measured" zenith distance $Z$ and the geodetic zenith distance $\overline{Z}$.

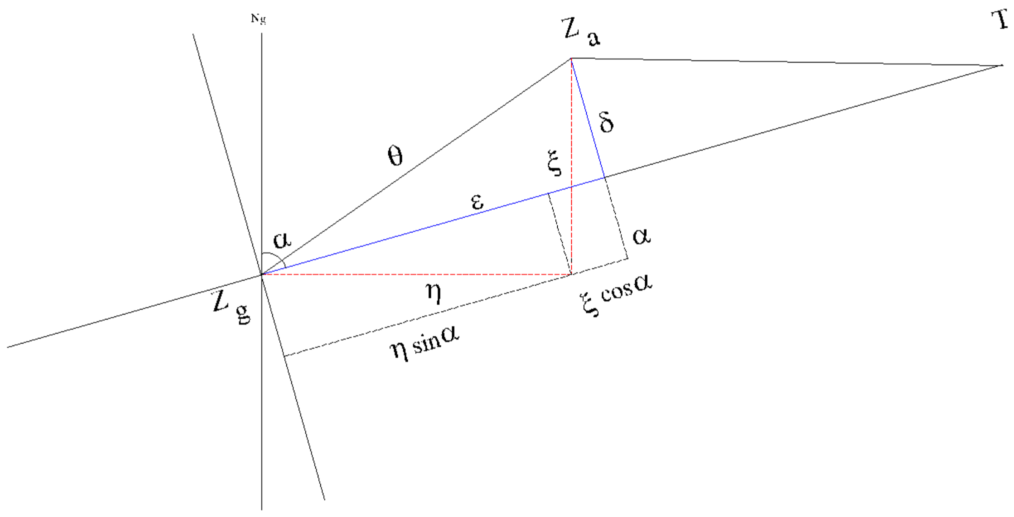



FIG 6-2

the relation between both zenith distances is simply given by;

$$Z = Z' + \varepsilon \tag{6-13}$$

## 6-3 Gravimetric Effect on Horizontal Angles

Any horizontal angle may be considered as the differences between two azimuths. Accordingly the corresponding horizontal angle, which refers to the direction of the normal "$W$", can be computed as the difference between the two corresponding geodetic azimuths:

$$W_{21} = \alpha_2 - \alpha_1 \tag{6-14}$$

where

$\alpha_1 = A_1 - \Delta\alpha_1$

$\alpha_2 = A_2 - \Delta\alpha_2$

Then, equation (6-14) takes the form

$$W_{21} = (A_2 - A_1) - (\Delta\alpha_2 - \Delta\alpha_1) \tag{6-15}$$

Hence we can apply equation (6-9) for $\Delta\alpha_1$ and $\Delta\alpha_2$. It can be seen that the main term ***η tan φ***drops out , so that equation (6-15) wil take the form;

$$W_{21} = (A_2 - A_1) - (\xi \sin\alpha_2 - \eta\cos\alpha_2)\cot Z_2 + (\xi\sin\alpha_1 - \eta\cos\alpha_1)\cot Z_1 \tag{6-16}$$

We can notice that for nearly horizontal lines of sight the difference both horizontal angles are very small and can be neglected.

## 6-4 Gravimetric Effect on Base Lines

Figure (6-3) illustrates the reduction of measured base lines to the ellipsoid. Denote an element of the measured distance by $d1$. It has an inclination $\beta$ towards the local horizon. The deflection component in the direction of the measured line that has the azimuth $\alpha$ is again denoted by $\varepsilon$. The element $ds$, which is the component of $d1$ parallel to the ellipsoid, is

$$ds = d1\cos(\beta - \varepsilon) = d1\cos\beta + \eta d1\sin\beta \tag{6-17}$$

Denoting the projection of d1 onto the local horizon by d1,

$$d1' = d1\cos\beta \tag{6-18}$$

And noting that

$$d1\sin\beta = dh \tag{6-19}$$

We have

$$ds = d1' + \varepsilon dh \qquad (6\text{-}20)$$

If R is the local radius of curvature of azimuth of the ellipsoid, then it is shown in differential geometry that

$$\frac{1}{R} = \frac{\cos^2\alpha}{M} + \frac{\sin^2\alpha}{N} \qquad (6\text{-}21)$$

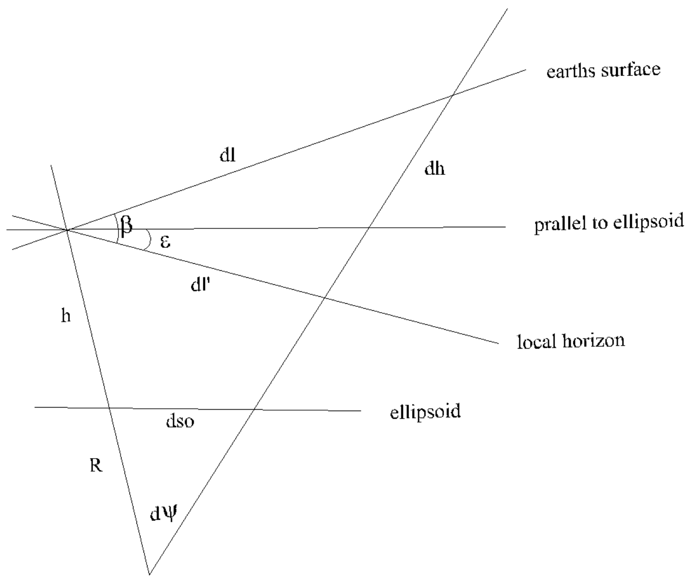



FIG 6-3

Where $M$ and $N$ are, respectively, the north-south and east-west radii of curvature. Then, if $ds_\circ$ is the projection of $d1$ onto the ellipsoid,

$$\frac{ds}{ds_\circ} = \frac{R+h}{R} = 1 + \frac{h}{R}$$

Or

$$ds_\circ = ds - \frac{h}{R} ds_\circ = d1 + \varepsilon dh - \frac{h}{R} ds_\circ \qquad (6\text{-}22)$$

Setting

$$ds_\circ / R = d\psi \qquad (6\text{-}23)$$

We have

$$ds_\circ = d1' + \varepsilon dh - hd\psi = d1' + d(\varepsilon h) - hd(\psi + \varepsilon)$$

And on integration between the end points A and B we get

$$S_{\circ} = d1' + \varepsilon_B h_B - \varepsilon_A h_A - \int_A^B h d(\psi + \varepsilon) \tag{6-24}$$

If the elevation $h$ is nearly constant along the line, as always occurs in base-line measurements, then the application of a mean-value theorem of integral calculus gives

$$S_{\circ} = l' + \varepsilon_B h_B - \varepsilon_A h_A - h_m(\varepsilon_B - \varepsilon_A)\int_A^B d\psi$$

Here

$$l' = \int_A^B d1 \cos\beta$$

Is the sum of the locally reduced $d1'$, and $h_m$ is a mean elevation along the line. On expressing $d\varphi$ in terms of $ds_{\circ}$ by (6-23) and integrating we finally obtain

$$S_{\circ} = l' + \varepsilon_B(h_B - h_m) - \varepsilon_A(h_A - h_m) - \frac{hm}{R} S_{\circ} \tag{6-25}$$

Strictly speaking, $R$ the local ellipsoidal radius of curvature of azimuth $\alpha$, is slightly variable along the line $A$ to $B$. In practice, however, it is perfectly permissible to replace the local value of $R$ by its average along the line, so that $R$ in (6-23) can be considered constant, which leads to (6-25). This amount of the approximation of the ellipsoidal arc AB by a circular arc whose radius R is the average along AB of the values given by (6-21).

The terms with $\varepsilon_A$ and $\varepsilon_B$ represent the effect of the inclination between the geopotential and spheropotential surfaces, they will often be negligible. The term ( $S_{\circ} h_m / R$) is due to the convergence of the ellipsoidal normals.

The rigorous reduction of base lines according to the equation (6-25) thus involves the geoid undulation $N$, through the height h above the ellipsoid, and the deflection of the vertical $\varepsilon$. The base lines are reduced directly to the ellipsoid by means of the straight ellipsoidal normals, in conformity with Helmert's projection.

## 6-5 Reduction of Spatial Distances

Electronic measurement of distance yields straight spatial distance $l$ between two points $A$ and $B$ (figure 6-4). These distances may either be used directly for computation in geodetic coordinate system $\phi$, $\lambda$, h as in three-dimensional geodesy, or they may be reduced to the surface of the ellipsoid to obtain chord distances $l_0$ or geodesic distances $S_{\circ}$. The ellipsoidal arc $A_{\circ}$ $B_{\circ}$ is approximated

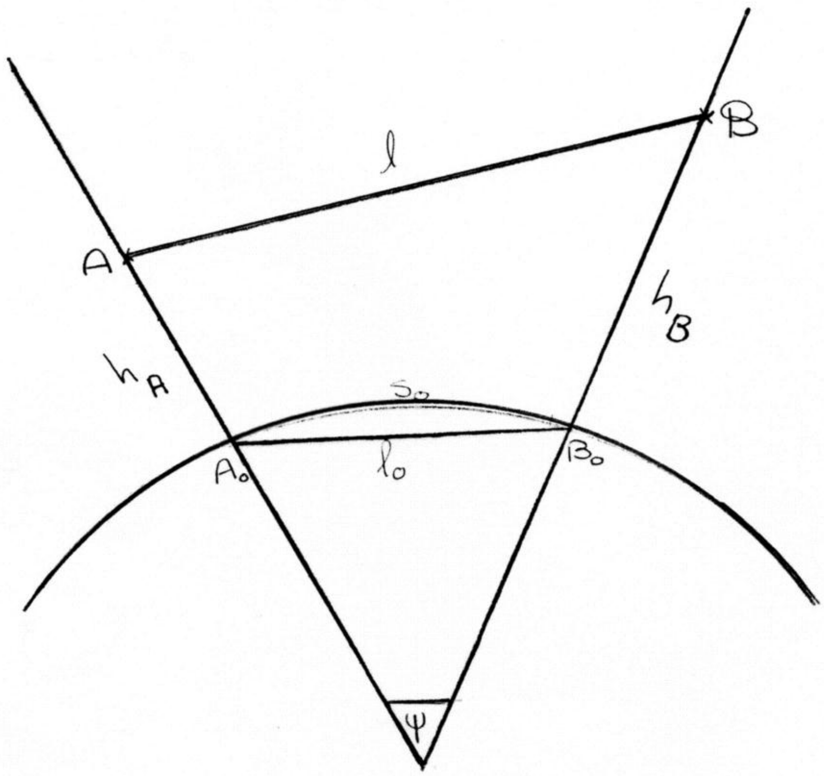



Fig. (6,4)

by circular arc of radius $R$ that is the mean ellipsoidal radius of curvature along $A_\circ B_\circ$ applying the law of cosines to the triangle OAB and we get

$$l^2 = (R+h_1)^2 + (R+h_2)^2 - 2(R+h_1)(R+h_2)\cos\psi$$

with $\cos\psi = 1 - 2\sin^2(\psi/2)$

This is transformed into

$$l^2 = (h_2 - h_1)^2 + 4R^2(1+\frac{h_1}{R})(1+\frac{h_2}{R})\sin^2(\psi/2)$$

and with

$$l_\circ = 2R\sin(\psi/2)$$

and the abbreviation $\Delta h = h_2 - h_1$ we obtain

$$l^2 = \Delta h^2 + (1+\frac{h_1}{R})(1+\frac{h_2}{R})l_\circ^2$$

Hence the chord $l$ and the arc $S_\circ$ are expressed by

$$l^2 = ((l-\Delta h^2)/(l+(h_1/R))(l+(h_2/R)))^{0.5}$$

$$S_\circ = R\psi = 2R\sin^{-1}(l_\circ/2R)$$

The reason for the great difference between the reduction procedures for base lines may be considered as measured along the earth's surface and piecewise reduced to the local horizon, which involves the direction of the vertical, whereas straight distances are independent of the vertical. Therefore, the deflection of the vertical does not have any influence on it.

# Chapter 7
# Coordinate Transformation and Datum Shift

## 7-1 Similarity Transformation

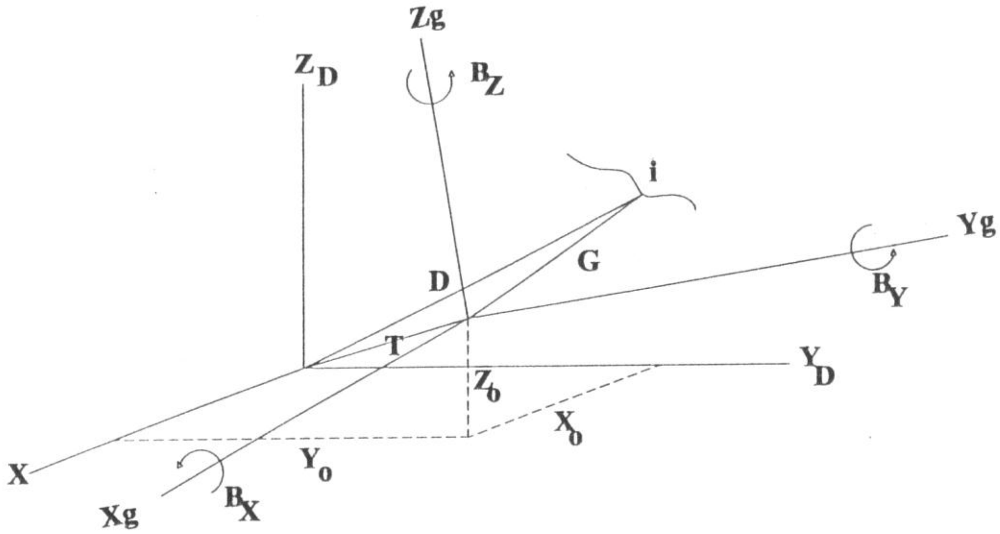
In geodesy several transformation models are used for the transformation from one system to another. Consider for example, the satellite Doppler coordinat system $D = [X_D, Y_D, Z_D]^T$ and the geodetic coordinate system $G = [X_G, Y_G, Z_G]^T$. The determined three-dimensional satellite Doppler coordinates are assumed to be the average terrestrial system. The average terrestrial system is a geocentric system. Also consider the geodetic system as the reference frame of the terrestrial network. Then the transformation between these two systems can be done by using several models, among them are the following models.

### 7-1-1 Bursa Model

Considering that the Doppler and terrestrial network position vectors $D_1$ and $G_1$ are observable, the transformation equation expressed in the Doppler system is readily seen from figure (7-1),

$$F = T + (1+\Delta)\cdot RG - D = 0 \qquad (7\text{-}1)$$

Where



- T denotes the translation vector between the origins of the two systems in the D-system.
- $1+\Delta$ denotes the scale factor between the systems
- R is the product of three consecutive orthogonal rotations around the axes of the G-system, and can be given as follows

$$R = R_Z(B_Z)\cdot R_Y(B_Y)\cdot R_X(B_X) \tag{7-2}$$

Using the conventional definitions of rotation matrices, one can write

$$R_{x_G}(B_x) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos B_x & \sin B_x \\ 0 & -\sin B_x & \cos B_x \end{vmatrix}$$

$$R_{y_G}(B_y) = \begin{vmatrix} \cos B_y & 0 & -\sin B_y \\ 0 & 1 & 0 \\ \sin B_y & 0 & \cos B_y \end{vmatrix}$$

$$R_{z_G}(B_z) = \begin{vmatrix} \cos B_z & \sin B_z & 0 \\ -\sin B_z & \cos B_z & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{7-3}$$

Since, the rotation between the two systems are small, it is permissible to write R in the following form.

$$R = I + r = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & B_z & -B_z \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{vmatrix} \tag{7-4}$$

Substituting (7-4) in (7-1) and neglecting second order terms in scale and rotations $B_x B_y B_z$ and their products, then the model can be written as

$$F = T + G + \Delta G + rG - D = 0 \tag{7-5}$$

Equations (7-2) and (7-5) define the relation between the two systems D and G in terms of seven parameters, three translation parameters, three rotation parameters, and scale factor. They are solved for in a least squares solution. The Cartesian coordinates of both systems are taken as observations. Equation (7-1) forms the mathematical model.

$$F(L_a, X_a) = 0 \tag{7-6}$$

Or

$$F(L_b+V, X_{\circ}+X)=0 \qquad (7\text{-}7)$$

where

$L_a$ denotes the adjusted observation
$X_a$ the adjusted parameters
$L_b$ the observations
$X_0$ the approximate parameters
V the residuals
X the parameters solved for

Each point $P_i$ yields one equation of (7-5). The model (7-5) can now be linearized and the usual adjustment procedure

$$V^T PV = \min \qquad (7\text{-}8)$$

Subject to the condition

$$BV + AX + W = 0 \qquad (7\text{-}9)$$

applied where

$B = \partial F / \partial L_a$, $A = \partial F / \partial X_a$, $W = F(L_b, X_0)$

And $P$ is the weight matrix. Each point contributes three equations to the equation system (7-9), e.g., for point $P_i$, taking $X_0$ = 0 one has

$$\underbrace{\left\langle\begin{array}{ccc|ccc}1&0&0&-1&0&0\\0&1&0&0&-1&0\\0&0&1&0&0&-1\end{array}\right\rangle}_{B_i}\cdot\underbrace{\begin{vmatrix}V_{XG}\\V_{YG}\\V_{ZG}\\V_{XD}\\V_{YD}\\V_{ZD}\end{vmatrix}}_{V_i}+\underbrace{\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}}_{\mathrm{I}}\underbrace{\begin{matrix}X_G&Y_G&-Z_G&0\\Y_G&-X_G&0&Z_G\\Z_G&0&X_G&-Y_G\end{matrix}}_{A}\underbrace{\begin{vmatrix}dX\\dY\\dZ\\\Delta\\B_Z\\B_Y\\B_X\end{vmatrix}}_{X}+\underbrace{\begin{vmatrix}X_G-X_D\\Y_G-Y_D\\Z_G-Z_D\end{vmatrix}}_{\mathrm{w_i}}=0$$

(7-10)

The solution of the above system is as follows:

$$X = -(A^T M^{-1} A)^{-1} A^T M^{-1} W \qquad (7\text{-}11)$$

$$V = -P^{-1} B^T M^{-1}(AX + W) \qquad (7\text{-}12)$$

$$M = BP^{-1}B^T \qquad (7\text{-}13)$$

$$\sigma_0^2 = (V^T PV)/DF \qquad (7\text{-}14)$$

## 7-1-2 Molodeniski-Badekas transformation model

The difference between this model and Bursa Model is that the scale and rotation parameters are interpreted as pertaining to the terrestrial network represented by observable $G_{0i}$ . The transformation equation for this model is easily given from figure (3-2) as follows

$$F = T + G_{\circ} + (1+\Delta) R G_{\circ i} - D_i = 0 \qquad (7\text{-}15)$$

Where

$G_{0i} = G_i - G_0$ = Terrestrial position vector differences

$G_0$ Is the position vector of the initial point o in the G-system

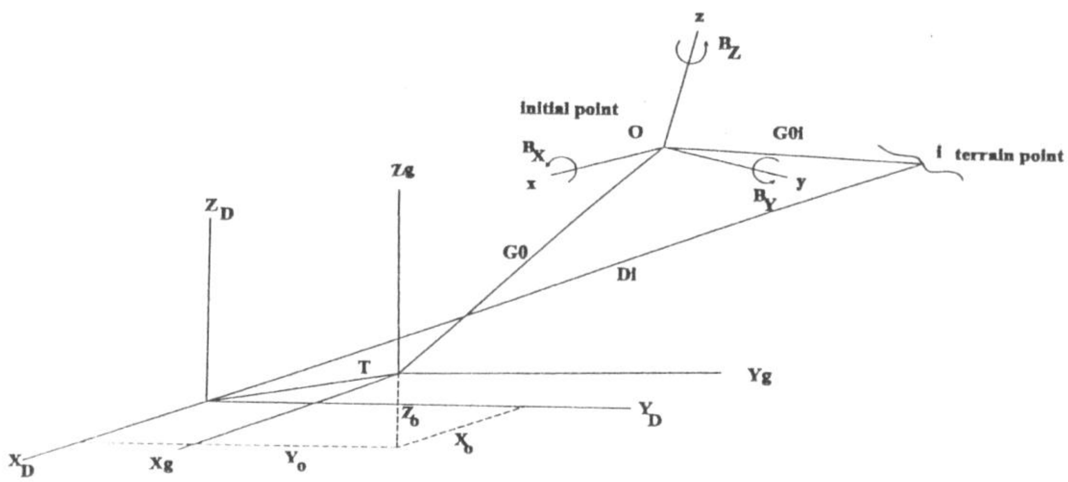
$G_i$ Is the position vector of the $i^{th}$ point from the initial point.

By neglecting the second order terms in scale and rotation we get the following form

$$F = T + G_{\circ} + \Delta G_{\circ i} + Q G_{\circ i} - D_i = 0 \qquad (7\text{-}16)$$

It can be noticed that in the Bursa model the geodetic position vector $G_i$ of each point is scaled and rotated, while with the Molodeniski-Badekas model only the interstation vectors $G_{0i}$ are scaled and rotated, and the position vector $G_0$ of the initial point 0 is not redefined.

Equation (7-16) is the mathematical model for least squares solution. Each common point contributes three condition equations similar to equation (7-10) and the solution of this system will follow the procedure of adjustment by the combined method.



## 7-2 Datum Shift

By means of transformation (5-5) we can compute the rectangular coordinates X, Y, Z from the geodetic coordinates $\varphi$, $\lambda$, h for a point outside the ellipsoid, which is defined by its dimensions, and its center, is the origin of the rectangular coordinate system. It is assumed that the center of this ellipsoid coincides with the earth center of gravity, that is, geocentric datum. Suppose that we define the same dimensions (semi-major axis a, and the flattening f) for another reference ellipsoid, its center does not coincide with the earth's center of gravity, but that the axis of the ellipsoid is parallel to the earth's axis of rotation. Let the coordinates of this center with respect to the rectangular coordinates system are $X_0$, $Y_0$, $Z_0$. Then the equations (5-5) must be modified so that they become

$$\begin{aligned} X &= X_O + (N + h)\cos\varphi\cos\lambda \\ Y &= Y_O + (N + h)\cos\varphi\sin\lambda \\ Z &= Z_O + ((b^2/a^2)N + h)\sin\varphi \end{aligned} \tag{7-17}$$

Where the principle radius of curvature in the prime vertical is known from geometry of the ellipsoid by

$$N = a(1-e^2\sin^2\varphi)^{-0.5} = a(1-f(2-f)\sin^2\varphi)^{-0.5}$$

Also the following approximation will be used letter in the differential formulas

$N = a(1+f\sin^2\varphi+\cdots)$, and

$$N(1-e^2) = N(1-2f+f^2) = a(1-2f+f\sin^2\varphi+\cdots)$$

which leads to

$$N \approx (1-e^2)N = a$$

If we vary the geodetic coordinates by small amount $\delta\varphi$, $\delta\lambda$, and $\delta$ h, and if we also alter the geodetic datum, reference ellipsoid, by $\delta a$, $\delta f$, and its position by small translation, parallel displacement, $\delta x_0$, $\delta y_0$, and $\delta z_0$. Then the rectangular coordinates X, Y, Z change by.

$$\begin{vmatrix}\delta X\\ \delta Y\\ \delta Z\end{vmatrix} = \begin{vmatrix}\delta X_o\\ \delta Y_o\\ \delta Z_o\end{vmatrix} + \begin{vmatrix}\partial X/\partial\varphi & \partial X/\partial\lambda & \partial X/\partial h\\ \partial Y/\partial\varphi & \partial Y/\partial\lambda & \partial Y/\partial h\\ \partial Z/\partial\varphi & \partial Z/\partial\lambda & \partial Z/\partial h\end{vmatrix} \cdot \begin{vmatrix}\delta\varphi\\ \delta\lambda\\ \delta h\end{vmatrix} + \begin{vmatrix}\partial X/\partial a & \partial X/\partial f\\ \partial Y/\partial a & \partial Y/\partial f\\ \partial Z/\partial a & \partial Z/\partial f\end{vmatrix} \cdot \begin{vmatrix}\delta a\\ \delta f\end{vmatrix} \tag{7-18}$$

$$\delta X = \delta X_o + R1 * \delta C + R2 * \delta E \tag{7-19}$$

And the partial derivatives will take the following form, (Soler 1976):

$\partial X/\partial\varphi = -a\sin\varphi\cos\lambda$, $\partial X/\partial\lambda = -a\cos\varphi\sin\lambda$, $\partial X/\partial h = \cos\varphi\cos\lambda$

$\partial Y/\partial\varphi = -a\sin\varphi\sin\lambda$, $\partial Y/\partial\lambda = a\cos\varphi\cos\lambda$, $\partial Y/\partial h = \cos\varphi\sin\lambda$

$\partial Z/\partial\varphi = a\cos\varphi$, $\partial Z/\partial\lambda = 0$, $\partial Z/\partial h = \sin\varphi$

$\partial X/\partial a = \cos\varphi\cos\lambda$, $\partial X/\partial f = a\sin^2\varphi\cos\varphi\cos\lambda$

$\partial Y/\partial a = \cos\varphi\sin\lambda$, $\partial Y/\partial f = a\sin^2\varphi\cos\varphi\sin\lambda$

$\partial Z/\partial a = \sin\varphi$, $\partial Z/\partial f = a\sin\varphi(\sin^2\varphi - 2)$

Considering that the position of the point in space remains unchanged, then $\delta X$ = 0, the change of the geodetic coordinates $\varphi$, $\lambda$, h can also be represented as a function of the variation in the geodetic datum (a, f, $x_0$, $y_0$, $z_0$), (Heiskanen & Moritz 1967), (Wolfgang 1980) as follows

$$R_1^{-1}\delta X_0 = -R_1^{-1}R_1\delta C - R_1^{-1}R_2\delta E \tag{7-20}$$

Then

$$\delta C = -R_1^{-1}\delta X_0 - R_1^{-1}R_2\delta E \tag{7-21}$$

$$\begin{vmatrix}\delta\varphi\\ \delta\lambda\\ \delta h\end{vmatrix} = \begin{vmatrix}\dfrac{\sin\varphi\cos\lambda}{a} & \dfrac{\sin\varphi\sin\lambda}{a} & \dfrac{-\cos\varphi}{a}\\ \dfrac{\sin\lambda}{a\cos\varphi} & \dfrac{-\cos\lambda}{a\cos\varphi} & 0\\ -\cos\varphi\cos\lambda & -\cos\varphi\sin\lambda & -\sin\varphi\end{vmatrix} \cdot \begin{vmatrix}\delta X_0\\ \delta Y_0\\ \delta Z_0\end{vmatrix} + \begin{vmatrix}0 & 2\cos\varphi\sin\varphi\\ 0 & 0\\ -1 & a\sin^2\varphi\end{vmatrix} \cdot \begin{vmatrix}\delta a\\ \delta f\end{vmatrix}$$

It is possible also to represent the change $\delta\varphi$, $\delta\lambda$, $\delta$ h, of the geodetic coordinates $\varphi$, $\lambda$, $h$ as a function of the variation $\delta\varphi_I$ ,$\delta\lambda_I$ ,$\delta h_I$ at the initial point $\varphi_I$ ,$\lambda_I$ ,$h_I$ instead of $\delta X_0$, $\delta Y_0$, $\delta Z_0$.

The translation vector $\delta X_0$ can be given in terms of the given $\delta\varphi_I$ ,$\delta\lambda_I$ ,$\delta h_I$ at the initial point by using equation (7-19) and setting

$$\delta X = 0, \ \delta C = \delta C_I, \qquad R_1 = R_{1I}, \qquad R_2 = R_{2I},$$

And after rearrange it will take the form

$$\delta X_0 = -R_{1I}\delta C_I - R_{2I}\delta E \qquad \text{7-22}$$

Now inserting equation (7-22) in (7-21) we get

$$\delta C = R_1^{-1}R_{1I}\delta C_I + (R_1^{-1}R_{2I} - R_1^{-1}R_2)\delta E \qquad \text{7-23}$$

Where

$$R_1^{-1}R_{1I} = \begin{vmatrix} \begin{matrix}\cos\varphi\cos\varphi_I + \\ \sin\varphi\sin\varphi_I\cos(\lambda-\lambda_I)\end{matrix} & -\sin\varphi\sin(\lambda-\lambda_I) & \begin{matrix}-[\sin\varphi\cos\varphi_I\cos(\lambda-\lambda_I) + \\ \sin\varphi_I\cos\varphi]/a\end{matrix} \\ \dfrac{\sin\varphi_I\sin(\lambda-\lambda_I)}{\cos\varphi} & \dfrac{\cos\varphi_I\cos(\lambda-\lambda_I)}{\cos\varphi} & \dfrac{-\cos\varphi_I\sin(\lambda-\lambda_I)}{a\cos\varphi} \\ \begin{matrix}a[\sin\varphi\cos\varphi_I - \\ \cos\varphi\sin\varphi_I\cos(\lambda-\lambda_I)]\end{matrix} & a[\cos\varphi\sin(\lambda-\lambda_I)] & \begin{matrix}\sin\varphi\sin\varphi_I + \\ \cos\varphi\cos\varphi_I\cos(\lambda-\lambda_I)\end{matrix} \end{vmatrix}$$

And

$$R_1^{-1}R_{2I} - R_1^{-1}R_2 = \begin{vmatrix} \begin{matrix}\cos\varphi\sin\varphi_I - \\ \sin\varphi\cos\varphi_I\cos(\lambda-\lambda_I)\end{matrix} & \begin{matrix}-a\sin\varphi\cos\varphi_I\sin^2\varphi_I\cos(\lambda-\lambda_I) + \\ \cos\varphi\sin^2\varphi_I - 2\cos\varphi\cos\varphi_I\end{matrix} \\ -\cos\varphi_I\sin(\lambda-\lambda_I) & -a\cos\varphi_I\sin^2\varphi_I\sin(\lambda-\lambda_I) \\ \begin{matrix}\sin\varphi\sin\varphi_I + \\ \cos\varphi\cos\varphi_I\cos(\lambda-\lambda_I)\end{matrix} & \begin{matrix}a[\cos\varphi\cos\varphi_I\sin^3\varphi_I + \\ \sin\varphi\sin^3\varphi_I - 2\sin\varphi\sin\varphi_I]\end{matrix} \end{vmatrix}$$

The final form of this equation expresses the variations $\delta\varphi$, $\delta\lambda$, $\delta$ h, at an arbitrary point in terms of the variations $\delta\varphi_I$ ,$\delta\lambda_I$ , $\delta h_I$ at the initial point, and also the changes $\delta a$ and $\delta f$ of the parameters of the ellipsoid.

Equation (7-23) can be expressed in terms of the variations of the deflection components $\zeta$ , $\eta$, and the geoid undulation N by substituting $\delta\varphi$, $\delta\lambda$, and $\delta$ h, by -$\delta\zeta$ , -$\delta\eta$ and $\delta$ N as follows

$$\delta\varphi = -\delta\xi \qquad \text{7-23a}$$

$$\delta\lambda\cos\varphi = -\delta\eta \qquad \text{7-23b}$$

$$\delta h = -\delta N \qquad \text{7-23c}$$

This is true because the astronomical coordinates are not affected by a datum shift and remain unchanged. These formulas for the effect of a shift of the geodetic datum are the well-known Vening Meinesz transformation formula.

## 7-3 Best Fitting Datum and How To Achieve It In Practice.

In our definition of a geodetic datum,  we have distinguished between global (or geocentric) datum and regional (local or geodetic) datum. The second one, although having its centre shifted generally from the earth's centre of mass, it does have the advantage of approximating the geoid as much as possible in the particular region of interest. In other words, we can state that the main objective of using a regional datum is to get minimum deviations between the geoid and reference ellipsoid over the area in question. When we achieve such an objective, we say that we have obtained a "best fitting ellipsoid" or a "best fitting datum", e.g., for our country. An ellipsoid that fits the geoid very well in a certain country does not necessarily fit in other country.

The problem of determining the datum positional parameters at the initial point, as discussed in the previous sections, is solved temporarily by assuming the ellipsoid and geoid to be tangent at the intial point as a preliminary orientation and use the astronomic observations to fix the other parameters at the initial point. Such a constraint does not, of course, provide a best fitting ellipsoid for our region, since the deviations (deflections and undulations) between the elliposid and the geoid may increase drastically as we go away from the initial point. In this case, the only region of best fitting would be a very limited area around the initial point. Therefore, we have to change the assumed (preliminary) positional parameters at the initial point, as well as the two parameters defining the size and shape of the reference ellipsoid, and repeat our caculation of the geodetic control network until we obtain the minimum deviations between of interest.

The above explained procedure is known in practice as the iterative solution to the problem of achieving a best fitting ellipsoid for a certain region. This proceus can be summarised in the following steps:

1. Select the position of the datum initial point "i" (starting point of the network) to be in the geometrical centre of the region of interest, and having a rigid terrain surrounded by areas of modest variations in gravity.
2. Select a reference ellipsoid, among the large list of ellipsoids used in practice, and specify the values of two parameters defining its size and shape (e.g. a and f).

3. Perform the preliminary orientation of the selected ellipsoid at the datum initial point, by setting : $\xi_i = \eta_i = N_i^* = 0$, and use the astronomic measurements to determine the geodetic coordinates of the initial point, as well as the geodetic azimuth of one initial line, i.e. = $\phi_i = \Phi_i$ , $\lambda_i = \Lambda_i$ and $\alpha_{ij} = A_{ij}$. By using equations (5,2a),(5,2b),(6,2)
4. Form the observation equations for directions,azimuthes and distances for the netweork (taking the appropriate weights of observations into account), and perform a least-squares rigorous adjustment ending-up with the adjusted values of the network coordinates $\varphi$ and $\lambda$.
5. Measure the astronomic latitude $\Phi_k$ and astronomic longitude $\Lambda_k$ at all points "k" of the network, i.e. k = 1,2, .... n where n is the number of points in the network. From these astronomic coordinates and their corresponding geodetic values obtained in step number (4) above, we can use equations (5 – 2a), (5-2b) and (5-3) to compute the astrogeodtic geoid, i.e. deflection components $\xi_k$ , $\eta_k$ and undulation $N_k^*$ - relative to the preliminary information at the initial point, where these quantities are fixed temporarily at zero values.
6. Select one of the conditions of minimizing the deviations between the reference ellipsoid and the geoid. These conditions can be set-up as follows:

$$\sum_{k=1}^{n} \bar{N}_k^* = 0 \; ; \tag{7-24}$$

$$\sum_{k=1}^{n} \bar{N}_k^{*\,2} = \text{minimum}, \tag{7-25}$$

$$\text{or :} \quad \sum_{k=1}^{n} \bar{\theta}_k^{\,2} = \text{minimum}, \tag{7-26}$$

in which the "bar" is assigned for each quantity after minimization and "θ" is the total deflection of the vertical. In practice, however, the last condition is the most popular, and hence it is usually the one being used. Equation (7-26) can be rewritten again, using eq. (5-2c), as :

$$\sum_{k=1}^{n} ( \xi^2{}_k+\eta^2{}_k ) = \text{minimum}. \tag{7-27}$$

7. Denote the observed (i.e. computed) astrogeodtic deflection components, obtained in step number (5) above, by just $\xi_k$ and k , i.e. without the "bar" used in step number (6), we can write the following expressions;

$$\xi_k^- = \xi_k + d\,\xi_k$$
$$\eta_k^- = \eta_k + d\,\xi_k \qquad (7\text{-}28)$$

where d $\xi_k$ and d $\eta_k$ are the changes required to be applied to the observed deflections ($\xi_k$ , $\eta_k$) to make the sum of their squares a minimum and provide a bestfitting ellipsoid, according to eq. (7-27). These changes, e.g. d$\xi_k$ , d $\eta_k$ can be expressed as a function whose main arguments are the required changes in the ellipsoid size and shape parameters (a, f) and the independant three positional parameters ($\xi_i$, $\eta_i$, $N_i^*$) which were incorrectly specified to be zeros at the datum initial point. Such a function can be simply expressed as follows:

$$d\xi_k = F_1\,(d\xi_i\,,\,d\eta_i\,,\,d\,N_i^*,\,da,\,df), \qquad (7\text{-}29)$$

and :

$$d\,\eta_k = F_2\,(d\xi_i\,,\,d\eta_i\,,\,d\,N_i^*,\,da,\,df), \qquad (7\text{-}30)$$

8. We can write these two equations for d$\xi_k$ and d $\eta_k$ For all stations having observed astrogeodetic deflections “k”, k = 1, 2 .....n, , in matrix notation as:

$$V = AX + L\,, \qquad (7\text{-}31)$$

where

V is the vector of deflection components after minimization.

L is the vector of astrogeodetic deflection components before minimization (i.e. computed from astro-observations.

X is a vector of five unknown components which are the two corrections to the chosen ellipsoid (da, df )and three corrections to the assumed deflection components and geoid undulation at the datum intial point (d$\xi_i$ , d$\eta_i$ , d $N_i^*$).

A is known as the cooficient matrix of the unknown parameters X.

9. Apply the parametric least-squares estimation procedure on eq. (7-31), which can be considered as an observation equation. The least-squares condition in this case will be:

$$V^TPV = \text{minimum}, \qquad (7\text{-}32)$$

where

P is the weight matrix of the observed astrogeodetic deflection components.

Equation (7-32) satisfies eq. (7-27) which is the condition for getting a best-fitting ellipsoid.

10. Substituting from (7-31) into (7-32) and perform the minimization process, we finally end-up with the following solution vector X of the required corrections to the five previously stated unknown parameters, which is :

    $X = (A^{T}PA)^{-1} (A^{T}PL).$ (7-33)

11. Add the compenents of X - vector, as obtoined from the last equation, to the assumed approximate values of the five parameters, and get the new best fitting values for ($\phi_i$, $\lambda_i$, $N_i$) of the initial point, as well as a,f .

# Chapter 8

## Classical and Three-Dimensional Geodetic Computations

### 8-1 Classical Geodetic Computation

All the observations in a triangulation network, which includes astronomical longitudes, latitudes, azimuths, horizontal directions, zenith distances, and spirit levelling, are gravity dependent. For example, all the angles observed by a theodolite are measured when it is leveled in such a way that its vertical axis lies in the direction of the gravity vector.

Since the computations are carried out on the surface of a reference ellipsoid, then we have to replace the actual observations by fictitious ones based on the direction of the normal to the ellipsoid.

The replacement of the actual, the corresponding ellipsoidal one knows observations by the fictitious ones, ellipsoidal, is known as the reduction of the actual observations. Consequently, measured angles are reduced to angles between two planes containing the ellipsoidal normal using equation (6-16), and zenith distances are reduced to the equivalent ellipsoidal one using equation (6-13). At stations where astronomical observations have been made Laplace equation is satisfied as in equation (6-9), distances measured between ground points are reduced to give distances between corresponding ellipsoidal points. It is important to notice that the reduction must be done through heights above ellipsoid.

After applying these reductions to all observations then a least-squares adjustment, either by using the method of correlates or by the variation of coordinates method, as described by many authors (e.g. Rainsford 1957, Tienstra 1965), is applied to the observations.

It is important to point out that the network is adjusted in a two dimensional frame consisting of latitudes and longitudes coordinate system defined on the surface of the reference ellipsoid. The observation point is assumed to lay on the normal to the ellipsoid through the adjusted geodetic position. The complete definition of this point with respect to the reference surface requires a third coordinate, which is the ellipsoidal height. This is obtained from a separate adjustment of the trignometrical levelling data observed for the network, this adjustment usually ignores the distinction between the elevations above mean sea level determined by spirit levelling which are more nearly orthometric heights, heights above geoid, and ellipsoidal heights, heights above the reference ellipsoid.

### 8-1-1 Major Defects

The major defects of the classical geodetic computation methods can be summarized as follows:

1) Forcing the horizontal angles to fit the Laplace azimuths may alter the preliminary positions enough to require a reassessment of the astrogeodetic deflections. The consequences of this may include changes in the Laplace azimuths, making necessary reiteration of the adjustment.
2) Distances reduced to the geoid may differ from the corresponding ellipsoidal distances, an error of 6 meters in the geoid separation results in an error of 1 p.p.m. in the base line. In a small country the difference may be small, but in a continental area it may be large especially if an old and ill-fitting ellipsoid is in use. An example of this case was found in Mergui base in India, where the base was 3 meters above mean sea level, and 100 meter above Everest spheroid and the acceptance of the mean sea level height would put it wrong by 1 in 60000. (Bomford 1971).
3) The corrected Laplace azimuths are burdened with the observational errors in astronomical latitudes, longitudes and azimuths, and accumulated errors in the geodetic survey between the origin and the observation point. Accordingly fixing these values during the adjustment procedure will affect the final adjusted values.
4) The observed astronomical latitudes and longitudes are not permitted to influence the final adjusted positions of the network stations.
5) The method cannot be adopted to permit the treatment of observed vertical angles and other levelling data simultaneously with the horizontal measurements.

## 8-2 Three Dimensional Geodesy

In three-dimensional geodesy all the observed quantities horizontal angles, distances, vertical angles, spirit levelling, astronomical latitudes, longitudes, and azimuths are combined in a single adjustment process, i.e. combining the horizontal and vertical adjustment of the network in one adjustment process. Accordingly, we can no longer deal with the two dimensional coordinate system as it will not provide us with a quite adequate reference frame for the adjustment and consequently, we need a three-dimensional Cartesian system of coordinates upon which all the computations and adjustments are related.

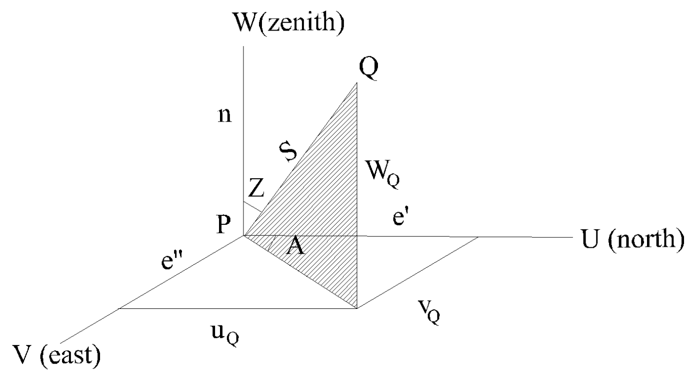



FIG 8-1

The transformation of the curvilinear coordinate values of point into a three-dimensional Cartesian system with its origin at the center of the ellipsoid, with one axis parallel to the rotational axis of the earth, and another laying in the meridian plane of Greenwich, is the well known equation (5-5). The relationships between such a coordinate system and the observed geodetic and astronomical observations in terrestrial survey networks have been the subject for study by geodesists over the past century.

Bruns 1878 established the basic and fundamental idea of the rigorous computations. The main observational data used were, horizontal angles, zenith distances, spatial distances, astronomical observations for latitudes, longitudes, and azimuths. Determining the position of any point $P$ requires five parameters, three of which are $X, Y, Z$ coordinate of the point based on a rectangular system, and the other parameters are the astronomical longitude and latitude of the point, which defines the direction of the plumb line. These observations are modeled in a local coordinate system $U, V, W$. The origin is at the observation station $P$, the $W$ axis coincides with the plumb line, while the $U$ and $V$ axes are pointing northward and eastward respectively, figure (8-1).

The azimuth A, and the zenith distances Z, to a neighbor station Q, distances S, from the observation station p. is given by

$$\tan A = {}^{v}\!/_{u}$$
$$\cos Z = {}^{w}\!/_{s} \qquad (8\text{-}1)$$
$$S = (u^2 + v^2 + w^2)^{0.5}$$

Taking $\Delta X$ as the vector leading from $P$ to $Q$ in a three coordinate system

$$\Delta X = \begin{vmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{vmatrix} = \begin{vmatrix} X_Q - X_P \\ Y_Q - Y_P \\ Z_Q - Z_P \end{vmatrix} \qquad (8\text{-}2)$$

Then

$$u = (\Delta X)^T e'$$
$$v = (\Delta X)^T e'' \qquad (8\text{-}3)$$
$$w = (\Delta X)^T n$$

Where, $e', e''$, and $n$ are the unit coordinate vectors in the UVW-system. The complete derivations for the relations between the local horizon system and the rectangular coordinate system are given in Chapter 5.

Inserting the values of $u, v, w$ from (8-3) into (8-1) we obtain

$$\tan A = \frac{-\Delta X \sin\Lambda + \Delta Y \cos\Lambda}{-\Delta X \sin\Phi \cos\Lambda - \Delta Y \sin\Phi \sin\Lambda + \Delta Z \cos\Phi}$$
$$\cos Z = \frac{\Delta X \cos\Phi \cos\Lambda + \Delta Y \cos\Phi \sin\Lambda + \Delta Z \sin\Lambda}{(\Delta X^2 + \Delta Y^2 + \Delta Z^2)^{0.5}} \qquad (8\text{-}4)$$
$$S = (\Delta X^2 + \Delta Y^2 + \Delta Z^2)^{0.5}$$

Since equation (8-4) is somewhat complicated, it’s often and convenient to assume suitable approximate values. To compute the corrections, one needs the following differential formulas:

$$
\begin{aligned}
\delta A &= a_1\delta X_P + a_2\delta Y_P + a_3\delta Z_P + a_4\delta X_Q + a_5\delta Y_Q + a_6\delta Z_Q + a_7\delta\Phi_P + a_8\delta\Lambda_P \\
\delta Z &= b_1\delta X_P + b_2\delta Y_P + b_3\delta Z_P + b_4\delta X_Q + b_5\delta Y_Q + b_6\delta Z_Q + b_7\delta\Phi_P + b_8\delta\Lambda_P \\
\delta S &= c_1\delta X_P + c_2\delta Y_P + c_3\delta Z_P + c_4\delta X_Q + c_5\delta Y_Q + c_6\delta Z_Q
\end{aligned}
\tag{8-5}
$$

The formulas stated above contain the main principles of three-dimensional geodesy. The correction can be done either in X, Y, Z system. The computations are done by taking for example , a preliminary values for $\phi, \lambda$, $h$ to be the same as $\Phi$,$\Lambda$, and H for the corresponding points. These preliminary coordinates are converted to rectangular coordinates X, Y, and Z according to the equation (5-5). Then azimuths, zenith distances and spatial distances are computed from these preliminary coordinates by means of (8-4), and compared with the corresponding observed values of A, Z, and S. Each difference will furnish one equation of type (8-5) where $\delta$ A, $\delta$ Z and $\delta$ S are the differences between the preliminary computed values and observed values.

A sufficient number of such an equation can then be solved by a least-squares adjustment procedure, for the unknowns $\delta\ \phi$, $\delta\ \lambda$, $\delta$ h, $\delta\ \Phi$, and $\delta\ \Lambda$. The parameters $\delta\ \Phi$, and $\delta\ \Lambda$ obtained by this process are really estimates of $\zeta$ and $\eta \sec\phi$, but are burdened by errors in the preliminary values of $\Phi$ and $\Lambda$ which are usually the observed values.

The methods, presented by Wolf 1963a, are based essentially on this method. In forming the coefficients, the constant terms of the observation equations, and the preliminary astronomical latitudes and longitudes were set to the corresponding most recent geodetic values.

Hotine 1969, suggests that if astronomical observations of $\Phi$ and $\Lambda$ have not been made at some station in the network, approximate values obtained by interpolating $\zeta$ and $\eta$ from other astrogeodetic stations in the area should be submitted.

Fubara 1972 adopted a similar model for the study of the requirements for successful three-dimensional adjustment, using first a simulated net of 15 stations and then a real 6 stations network. He concluded that

1) Precise astronomical latitudes should be observed at each station on the net.
2) No significant differences between the adjustment in curvilinear and Cartesian coordinates.
3) Because of the low precision on the vertical angles used, the geodetic heights were the most weakly determined. The remedial efficiency of including spirit leveled heights, and gravity observations were recognized and are being investigated.

Stolz 1972 used gravimetrically determined deflection of the vertical only at points where astronomical observations of latitudes and longitudes had not been observed.

The importance which Hotine, Fubara, and Stolz on having observed or interpolated astronomical latitudes at every station is due to the fact that the diagonal coefficient of $\delta\Phi$ in the normal equations matrix becomes very small if the astronomical latitude $\Phi$ is not available. The solution of a normal equation set with a number of small diagonal coefficients exhibits all the symptoms of ill conditioning.

Hradilek 1972 investigates the erroneous influence of refraction in the three-dimensional model and the possibility of estimating it from the vertical angles. Also the adjustment procedure, which should represent an optimal transfer of information from the original observables to their functions, to the coordinates of the points in particular. He pointed out also that except in the plain regions with the elevations under 200 meters above the sea level. A constant value of the coefficient of refraction for all lines of sight radiating from one station, but different for the other stations, proved to be most practicable.

## 8-3 Differential Formulas in Three-Dimensional Adjustment

For the purpose of adjustment, the method of variation of coordinates is a suitable one. The observation equations used for the adjustment are derived from equation (8-4) after linearizing it. A sufficient number of these observation equations preferably subjected to a least squares adjustment will permit the solution for the unknowns $\delta$X, $\delta$Y, $\delta$Z, $\delta\Phi$ and $\delta\Lambda$ for each station. The direct approach of evaluating the observation equations was given by Wolf 1963. Another approach for the evaluation of the differential formulas is given here (Moritz 1978).

It was mentioned before, section (5-5), that the coordinates of any point Q in a local system with respect to the observables S, Z, and A could be computed from the equation (5-1) and in the same time the relation connecting the local horizon system with X, Y, and Z system, section (5-6-3) was given by

$$U = R \cdot X \tag{8-6}$$

Where $R = \begin{vmatrix} -\sin\Phi\cos\Lambda & -\sin\Phi\sin\Lambda & \cos\Phi \\ -\sin\Lambda & \cos\Lambda & 0 \\ \cos\Phi\cos\Lambda & \cos\Phi\sin\Lambda & \sin\Phi \end{vmatrix}$

Differentiating equation (5-1) yields to

$$dU = D\begin{vmatrix} s & dz \\ s & \sin z & dA \\ & ds \end{vmatrix} \tag{8-7}$$

Where

$$D = \begin{vmatrix} \cos z\cos\alpha & -\sin\alpha & \sin z\cos\alpha \\ \cos z\sin\alpha & \cos\alpha & \sin z\sin\alpha \\ -\sin z & 0 & \cos z \end{vmatrix}$$

The quantities $\alpha, z, s$ are the normal equivalents of the observables $A, z, s$, so that

$$\begin{aligned} A &= \alpha + dA \\ Z &= z + dZ \\ S &= s + dS \end{aligned} \tag{8-8}$$

And can be computed from (8-4) by using approximate coordinates $X, Y, Z$ and replacing $\Phi, \Lambda$ by $\phi, \lambda$.

Now, by differentiating (8-6), and substituting $dU$ by its equivalent from (8-7), we get:

$$\begin{vmatrix} s & dz \\ s & \sin z & dA \\ & ds \end{vmatrix} = D^T R d\Delta X + D^T dR\Delta X \tag{8-9}$$

The second term, $D^T Rd\Delta X$, represents the effect of $d\Phi$ and $d\Lambda$, on the observable $Z, A, S$

$$dZ = \xi\cos\alpha + \eta\sin\alpha \tag{8-10}$$

$$dA = \xi\sin\alpha\cot z + \eta(\tan\phi - \cos\alpha\cot z)$$

The effect of $d\Phi$ and $d\Lambda$ on the spatial distance $S$ is zero, because $S$ doesn't depend on the direction of the plumb line.If we substitute $\xi$ by $\delta\phi$, and $\eta$ by $\delta\Lambda\cos\phi$ and rewrite the equation (8-9) in full, we get

$$\begin{vmatrix} \delta Z \\ \delta A \\ \delta S \end{vmatrix} = \begin{vmatrix} \dfrac{\Delta X\cos Z}{s^2\sin z} & \dfrac{\Delta Y\cos Z - s\cos\Phi\sin\Lambda}{s^2\sin z} & \dfrac{\Delta Z\cos z - s\sin\Phi}{s^2\sin z} \\ \dfrac{\sin\alpha\sin\Phi\cos\Lambda - \cos\alpha\sin\Lambda}{s\sin z} & \dfrac{\sin\alpha\sin\Phi\sin\Lambda + \cos\alpha\cos\Lambda}{s\sin z} & \dfrac{-\sin\alpha\cos\Phi}{s\sin z} \\ \dfrac{\Delta X}{s} & \dfrac{\Delta Y}{s} & \dfrac{\Delta Z}{s} \end{vmatrix}.$$

(8-11)

Instead of computing the corrections in the rectangular coordinate system X, Y, and Z as in (8-11), it is suitable to compute the corrections in the geodetic coordinate system $\phi, \lambda, h$, because the coordinates of the Egyptian geodetic network are given in this system. The transformation from the rectangular system to the geodetic system can be obtained by rewriting equation (7-19) with $\delta X_0, \delta Y_0, \delta Z_0, \delta a, \delta f = 0$

$$\begin{vmatrix} dX_Q - dX_P \\ dY_Q - dY_P \\ dZ_Q - dZ_P \end{vmatrix} + \begin{vmatrix} -\cos\alpha & -\cos\phi\sin\alpha & 0 \\ \cot z\sin\alpha & -\cos\alpha\cos\phi\cot z & 0 \\ 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \delta\Phi \\ \delta\Lambda \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} \delta X \\ \delta Y \\ \delta Z \end{vmatrix} = \begin{vmatrix} -\sin\phi\cos\lambda & -\sin\lambda & \cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & \cos\phi\sin\lambda \\ \cos\phi & 0 & \sin\phi \end{vmatrix} \cdot \begin{vmatrix} (N+h)\delta\phi \\ (N+h)\cos\delta A \\ \delta h \end{vmatrix}$$

(8-12)

Now, inserting the equation (8-12) into equation (8-11) and after some arrangement, we can get the required differential formula in the geodetic system $\phi$, $\lambda$, and $h$.

# Chapter 9
# Geodetic Positioning By Satellites Geodesy

## 9-1 Introduction

Artificial satellites are of use to geodesy in two ways, dynamic and geometrical. The dynamic is based on the fact that given the external gravity field of the earth, the orbit of the satellite is predictable.

Conversely, if the orbit is observed, certain constants in the expression of the field are determinate. The dynamic use may namely be for, Refinement of the gravity field, Geoid determination, polar motion determination, and Crustal movement.

In the geometrical use, the satellite is simply used as a beacon or radio transponder. Its movement is no more than nuisance, except in so far as it moves the beacon from one convenient site to another. Knowledge of the orbit is only required for predicting its movements. The geometrical use may take the form of satellite triangulation or trilateration or combination of the two or the Doppler method, and lastly GPS.

## 9-2 Kinds of artificial satellites

Earth satellite systems have been developed and operated successfully to provide a variety of services. Among these services we have, communication satellites, metrological, earth resources, navigational, military and geodetic satellites. For geodetic purposes the satellites are these kinds:

1) **Passive Satellite**, which does not have an internal power source and thus is illuminated by the sun, so it can be photographed against the celestial background.
2) **Co-operative Satellite**, which carries a passive target as a reflector that co-operate with a ground based emitter of power (as a laser beam) to reflect the emitted signal back to its source on the ground.
3) **Active Satellite**, which emits an optical or electronic signals using its own internal source of power and these signals are received by the instruments of the observing stations on the ground.

According to the characteristics of each kind of geodetic satellite, certain techniques of satellite observations will be required.

## 9-3 Orbits of satellites geodesy

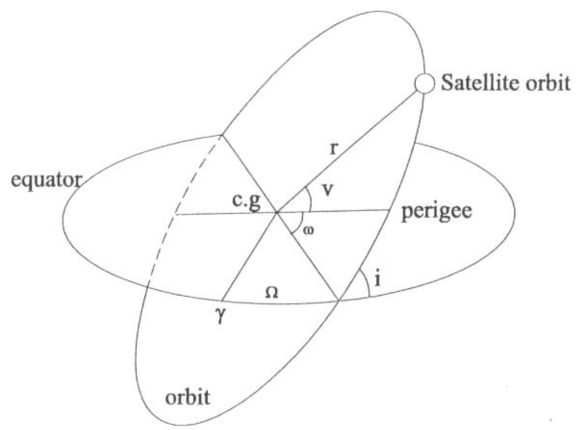
Considering a satellite moving around an imaginary uniform spherical earth, isolated in space with evenly distributed mass and with no atmosphere. Then the satellite orbit would be perfect Keplerian ellipse and would not change with time, with the earth's center at one of its foci.

Since the earth is not a uniform sphere, there are differences that cause an obvious change in the numerical values of the Keplerian orbital parameters. Nevertheless, it is convenient to regard the orbit to be, in the first approximation, Keplerian, and treat the perturbation as temporal variations of the six elements describing such a Keplerian motion.

Thus, these orbital elements will be functions of time and have to be updated at every epoch of observation.

The six Keplerian orbital parameters figure (9-1), are defined as follows;

1) The semi-major axis of the orbital ellipse "a"
2) The eccentricity of the orbital ellipse "e"
3) The inclination of the orbital plane with respect to the earth's equatorial plane "I"
4) The right ascension of the ascending node of the orbit "Ω"
5) The argument of perigee "ω"
6) The true anomaly "v"




Fig(9-1)

The first two parameters defines the size and shape of the orbital ellipse, the third and forth define the orientation of the orbital plane with respect to the earth's equator, the fifth parameter defines the orientation of the line of apsides in orbital plane, the sixth parameter defines the position of the satellite in the orbit at any epoch. The line of nodes is the intersection of the orbital plane with the plane of the equator; it connects the ascending and descending nodes. The right ascension of the node, Ω, is the angle between the line of nodes and the direction to the vernal equinox γ. The major axis of the orbit, known as line of apsides, intersects the orbital ellipse at the perigee, the position where the satellite is closest to the earth, and at the apogee, where the satellite is farthest away. The angle, ω, between the nodes and the major axis is the argument of perigee. The angular distances of the satellite S is called the true anomaly and denoted by V.

## 9-4 Satellite orbital coordinate system

The origin of the satellite orbital coordinate system is at the earth's center of mass "c.g". The X-axis coincide with the line of apsides, the Y-axis corresponds to V=90, and the Z-axis completes the right-handed system. From figure (9-2) the eccentric anomaly E is the angle between line of apsides and the line joining the geometrical center of the ellipse with the projection of the satellite S on the concentric circle of radius **a**. the relation between the true and eccentric anomaly is given as

$$\tan V = (1-e^2)^{0.5}\sin E / (\cos E - e) \qquad (9\text{-}1)$$

The instantaneous position vector of a satellite in the orbital system is given by:

$$r^{OR} = \begin{vmatrix} X \\ Y \\ Z \end{vmatrix}^{OR} = r\begin{vmatrix} \cos V \\ \sin V \\ 0 \end{vmatrix} = \begin{vmatrix} a(\cos E - e) \\ a(1-e^2)^{0.5}\sin E \\ 0 \end{vmatrix} \qquad (9\text{-}2)$$

The equation of the orbital ellipse may be written as

$$r = P / (1 + e\cos V) \qquad (9\text{-}3)$$

Where $r$ is the distance of the satellite from the earth's center of mass and

$$P = b^2 / a = a(1-e^2) \qquad (9\text{-}4)$$

Is the length of the radius vector r for V=90. Accordingly equation (9-2) can be given as a function of the true eccentric anomaly.

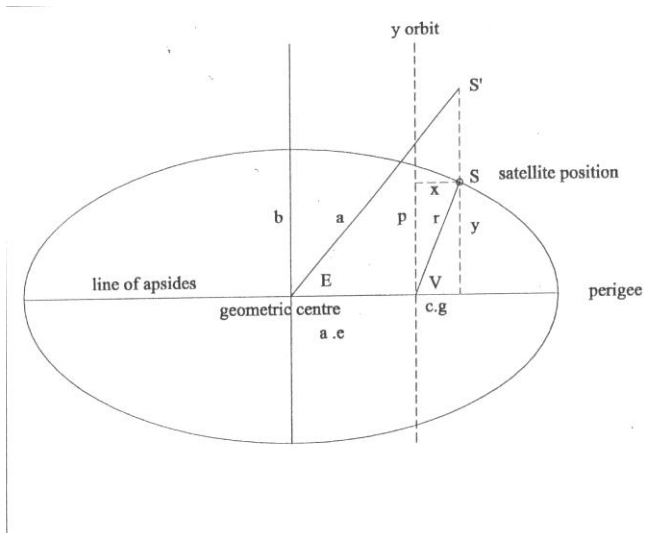


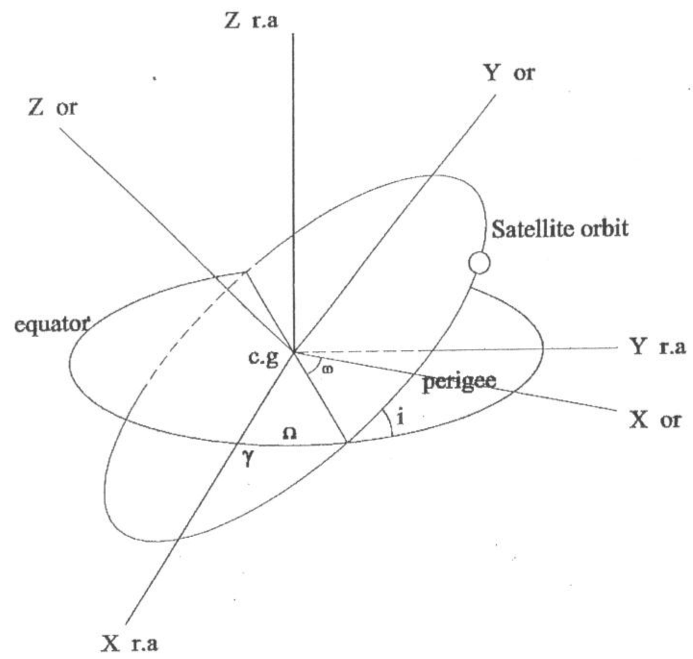

Fig (9-2)

## 9-5 Relation between satellite orbital coordinates system *OR* and the right ascension coordinate system *R.A*

The origin of the right ascension coordinate system is at the center of the gravity of the earth. The X-axis points towards the mean vernal equinox, and lies in the mean celestial equator. The Z-axis coincides with the mean rotational axis of the earth. The Y-axis is selected to make the system right-handed. The transformation from the orbital system (OR) to the RA system is done through three angles ω, Ω, figure (9-3).

The first rotation R (-ω) around Z brings the X axis in the equatorial plane; the second rotation R (- ί) around the new position of the X makes the Z-axis coincide with the mean rotational axis of the earth Z the final rotation R(-Ω) around the Z again makes the X coincide with the vernal equinox and the complete transformation reads

$$\begin{matrix} X \\ Y \\ Z \end{matrix}_{RA} = R_{Z^{OR}}(-\Omega) \quad R_{X^{OR}}(-i) \quad R_{Z^{OR}}(-\omega) \quad \begin{matrix} X \\ Y \\ Z \end{matrix}_{OR} \qquad (9\text{-}5)$$




Fig(9-3)

## 9-6 Relation between right ascension system R.A and average terrestrial system A.T

The right ascension system is transformed into the average terrestrial system by means of one single rotational angle (GAST) around the Z-axis, which makes the vernal equinox coincides with the Greenwich mean astronomical meridian, figure (9-4), the transformation from the R.A system to the A.T system is given as

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{AT} = R_Z^{RA}\,(GAST) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{RA} \qquad (9\text{-}6)$$

Combining equations (9-2), (9-5), (9-6) then the transformation of the satellite orbital coordinate system O.R to the average terrestrial system A.T is given as

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{AT} = R_Z^{RA}\,(GAST)\quad R_Z^{OR}(-\Omega)\quad R_X^{OR}(-i)\;\; R_Z^{OR}(-\Omega) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{OR} \qquad (9\text{-}7)$$

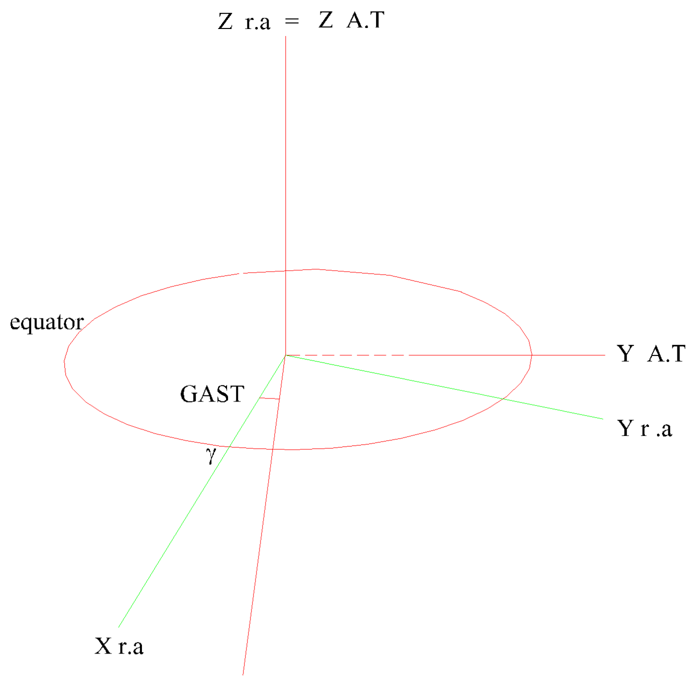


Fig 9-4

## 9-7 Satellite observation techniques

Since the satellite at a certain time is nothing else but an elevated target. Three different techniques of observation are usually used.

1. *Direction (or angle) measurements*; where simultaneous observations from ground stations to the satellite at a certain position will form a satellite triangulation.
2. *Range measurements*; it is obtained by timing the travel time of the electromagnetic waves "laser" between the tracking station and the satellite, or a radio pulse emitted usually by satellite and received at the tracking station. Thus forming a satellite trialteration.
3. *Range difference*; it is known in practice as Doppler measurements, which is analogous to the terrestrial observation of elevation differences.

### 9-7-1 Direction Measurements

For geodetic purposes the direction of a satellite at any instant cannot usefully be recorded by measuring its azimuth and zenith distances with a theodolite. The practical method is to photograph the satellite against a background of stars. The stars and satellite positions are identified on the photographic plate, and measurements are made for the location of the satellite image relative to images of the known stars. With the right ascension $\alpha$ and declination $\delta$ of the stars known, the topocentric $\alpha_{ij}$ and $\delta_{ig}$ of the satellite can be estimated. The mathematical model of this mode is given by using figure (9-5) as:

$$\boldsymbol{r}_j - \boldsymbol{r}_i - \boldsymbol{r}_{ij} = 0 \tag{9-8}$$

Where $r_i$ and $r_j$ are radius vectors of the observing point and satellite

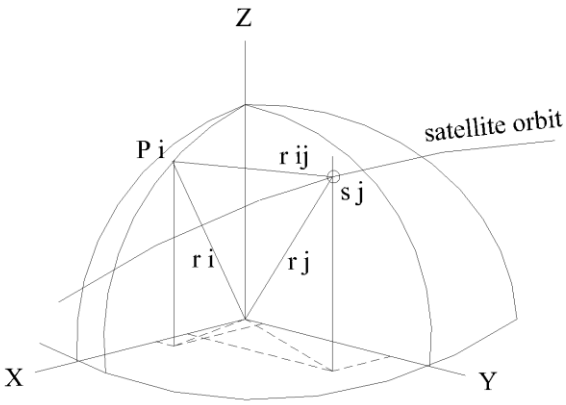



Fig 9-5

respectively.

The topocentric vector $r_{ij}$ in the A.T system is given by

$$r_{ij} = \begin{vmatrix} \delta X_{ij} \\ \delta Y_{ij} \\ \delta Z_{ij} \end{vmatrix} = R_{\underset{Z}{RA}}(GAST)r_{ij} \begin{vmatrix} \cos\delta_{ig}\cos\alpha_{ij} \\ \cos\delta_{ij}\sin\alpha_{ij} \\ \sin\delta_{ij} \end{vmatrix} \tag{9-9}$$

It should be mentioned that the orbit in this case is unknown. An astro-triangulation net can be formed by observing each satellite position from two ground stations at least. Also at least one measured distance or the coordinates of two ground stations are required for the scale. This simple procedure has several problems that limit its accuracy and make it complicated. The most troublesome is the uncertainty in modeling the camera lens distortion, atmospheric refraction and the accuracy of the catalogued star positions presents additional limitations.

### 9-7-2 Range measurements

This technique of observation is similar to trialteration technique where as in figure (9-5), $R_{ij}$ is the measured range, distance, from the tracking station Pi to the satellite position . The mathematical model can evidently be written as;

$$[(X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2]^{0.5} - r_{ij} = 0 \qquad (9\text{-}10)$$

From the equation (9-10) we can see that we need to know the coordinates of at least three satellites positions in order to solve three equations of the form (9-10) to obtain the coordinates of the unknown tracking station. On the other hand if the coordinates of at least three tracking stations are known, then by simultaneous range observations from these stations to the satellite instantaneous position, three equations of the form (9-10) can be solved for the unknown coordinates of the satellite position. From the above discussion we conclude that in the range measurements we need at least three known ground tracking coordinates for simultaneous observations with the other unknown stations.

### 9-7-3 Range difference measurements (Doppler technique)

The idea of the Doppler technique is defined as the apparent change in frequency of a transmitted signal due to the relative motion between the transmitter and receiver. The principle was discovered by Christian Doppler, an Austrian physicist, in 1842. In experimenting with sources of wave motion, which can be sound or light, he found that when an object emitting a signal at a constant frequency moves towards an observer, the observer received a signal, which is of higher frequency than the signal radiated. In short, any movement between a constant frequency source and an observer produced a change in the frequency of the received signal. This effect is referred to as Doppler frequency shift. This means that a greater number of pulses (higher frequency) will be received in a given time interval as the emitter approaches than will be received in the same length interval as the emitter departs.

In the Doppler method, the satellite is the transmitter and the Doppler receiver is the observer. The satellite transmits a highly stable electromagnetic signal at 400 megahertz, MHz, 400 million cycles per seconds, A signal is also transmitted at 150 MHz, which is used to correct for ionospheric refraction. To the receiver on the ground, the frequency of the 400 MHz signal appears to change as the satellite passes overhead. Figure (9-6) illustrates the Doppler measurement technique. The frequency *f*

$_r$ being received from the satellite consists of the frequency transmitted $f_t$ plus a Doppler frequency shift of up to $\pm 8$ kHz due to relative motion between the satellite and the receiver. The navigation receiver is equipped with a stable reference oscillator from which a 400 MHz ground reference frequency $f_g$ is derived.

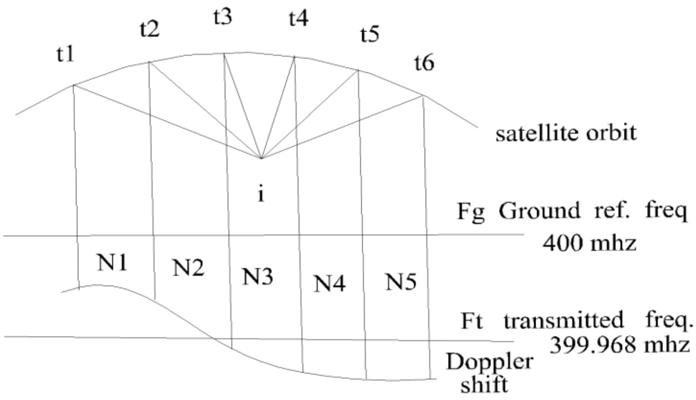



Fig (9-6)

As the satellite moves closer, more cycles per second must be received than were transmitted to account for the shrinking number of wavelengths along the propagation path. For each wavelength the satellite moves closer, one additional cycle must be received. Therefore, the Doppler frequency count is a direct measure of the change in distance between the receiver and the satellite over the Doppler count interval. In other words, the Doppler count is a geometric measure of range difference between the observer and the satellite at two points in space, accurately defined by the navigation message. This is a very sensitive measure because each count represents wavelength, which at 400MHz is only 0.75 meter.

The equation defining the Doppler count N of $f_g - f_r$ is the integral between respect of time marks from the satellite. For example,

$$N_1 = \int_{t1+r1/c}^{t2+r2/c} (f_g - f_r)dt \tag{9-11}$$

Note that $t_1 + r_1/c$ is the time of receipt of the satellite time mark that was transmitted at time $t_1$. The signal is received after propagation over distance $r_1$ at the velocity of light $c$. Equation (9-11) represents the actual measurements made by the satellite receiver, but it is helpful to expand this equation into two parts:

$$N_1 = \int_{t1+r1/c}^{t2+r2/c} f_g \ dt - \int_{t1+r1/c}^{t2+r2/c} f_r \ dt \tag{9-12}$$

Because the first integral in equation (9-12) is of a constant frequency $f_g$, it is easy for integration but the second integral is of a changing frequency $f_r$. However, the second integral represents the number of cycles received between the times of receipt of two timing marks. By a conservation of cycles argument this quantity must equal identically the number of cycles transmitted during the time interval between transmission of these time marks. Using this identity, equation (9-12) can be written

$$N_1 = \int_{t1+r1/c}^{t2+r2/c} f_g \ dt - \int_{t1}^{t2} f_t \ dt \tag{9-13}$$

Because the frequencies $f_g$ and $f_t$ are assumed constant during a satellite pass, the integrals in equation (9-13) become trivial, resulting in

$$N_1 = f_g[(t_2 - t_1) + \frac{1}{C}(r_2 - r_1)] - f_t(t_2 - t_1) \tag{9-14}$$

Rearranging the terms in equation (9-14)

$$N_1 = (f_g - f_t)(t_2 - t_1) + \frac{f_g}{C}(r_2 - r_1) \tag{9-15}$$

The values of the constants in equation (9-15) are

$t_2 - t_1 = t = 120$ Seconds.

$f_g = 400$ MHz

$C = 299,792.5$ Km/Second

And can be summarized as follows

- The observed quantity $N_1$ is an integer called Doppler Count.
- The frequency difference $(f_g - f_t)$ is assumed constant.
- The distance difference $(r_2 - r_1)$ is unknown and can be expressed in terms of the observed Doppler count as

$$\Delta r_{ijk} = r_2 - r_1 = \frac{C}{f_g}(N_1(f_g - f_t)(t_2 - t_1)) \tag{9-16}$$

With the satellite position known, the range difference can be used to determine the geodetic position of the receiver. From figure (9-7), the range difference mathematical model is given by

$$r_k - r_i - r_j - r_i - \Delta r_{ijk} = 0 \tag{9-17}$$

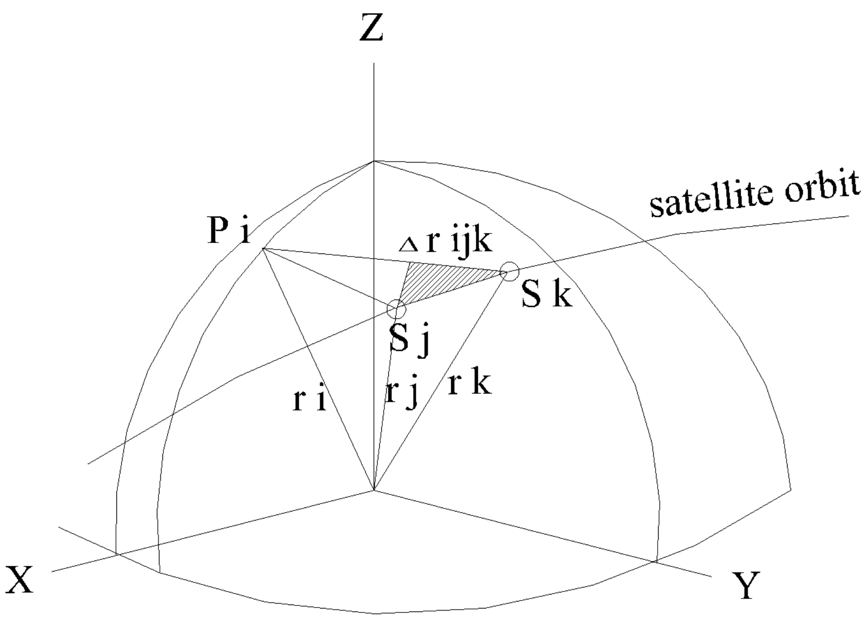



FIG 10-7

Where $r_1$ is the unknown position of the tracking station,$P_\mathrm{i}$ and $r_j, r_k$ are two known successive positions $S_j, S_k$, of the satellite on one orbital arc (pass) and $\Delta r_{ijk}$ is the range difference observed at $P_i$ between the two positions $S_j, S_k$. For each range difference measurements one equation of the form (9-17) can be written. Thus in theory, with three measurements the three unknown coordinates (x, y, z) can be uniquely determined, more than three observations lead to redundancy. In practice, care must be taken to include observations to satellite during different passes and with varied elevation angles so as to have a favorable geometrical configuration, otherwise, the configuration may lead to ill-conditioned solutions.

## 9-8 Satellite Signal "message"

Three kinds of information are transmitted by the active satellite,

1- Two frequency levels, 400 MHz and 150 MHz, controlled by a high stable oscillator, from which Doppler measurements can be made.
2- Accurate timing signals, two minute UT apart.
3- Orbital parameters, giving the predicted satellite position at instants of transmitting the timing signals.

The last two kinds are known as ephemeris, and it is defined as an ephemeris is basically a set of numbers or parameters which describe a satellite's orbit and its position along that orbit at a given time. There are two different types of ephemeris, Broadcast ephemeris and precise ephemeris.

### 9-8-1 Broadcast ephemeris System

The broadcast ephemeris is predicted by extrapolating tracking data collected by four stations, all of which lie in the united states territory. They are operated by the Navy Astronauts Group (NAG). There stations track all transit satellites at every opportunity and transmit the data to a central computing facility and control center. The orbit computation is carried out once per day . From the orbit determination, the broadcast ephemeris is extrapolated about 36 hours ahead. This computation is carried out daily,  with 24 hours of new data added each time. The predicted ephemeris is then injected into the respective satellite twice daily to maintain fresh orbital parameters.

### 9-8-2 Precise Ephemeris System

All the transit satellites are tracked using a network of over 20 tracking stations distributed around the world, and the tracking information is collected at a central computing facility. The precise ephemeris is computed for one or two of the satellites and the computations is carried out on alternate days using 48 hours of tracked data.

Distribution of the precise ephemeris is controlled and it is available only to the government agencies and not for commercial users. The ephemeris is generally considered accurate within 2 or 3 meters, but as it is available for more than two of the transit satellite (5 as total), its use involve relatively long occupancy of a station for acquisition of a prescribed number of passes.

Major differences between the precise and broadcast ephemeris systems can be summarized as follows:

1) The broadcast ephemeris is available at the time of observation, whereas the precise ephemeris, if it is obtainable at all, is not available until several weeks after observations has been taken.
2) The broadcast ephemeris is based on tracking information from 4 stations as opposed to about 20 for precise.
3) The broadcast ephemeris is a prediction of the satellite or orbit, where the precise ephemeris is derived from the observed orbit.
4) The two systems are based on different geopotential models and station coordinate sets and, in some cases, use different values for basic constants in the respective generating programs.
5) The broadcast ephemeris has an accuracy of about 20 to 30m, while the accuracy of the precise ephemeris is one order better at about 2 to 3m.

## 9-9 Different techniques of Doppler positioning

There are a number of ways Doppler positioning may be used to find the three dimensional position of a point on the earth's surface, the different techniques may be classified according to the type of position obtained, absolute or relative position. Point positioning techniques will give an absolute position of a point. Translocation or short arc techniques will give the position of points relative to other points.

### 9-9-1 Point positioning

Point positioning system is the process of collecting data from multiple satellite passes at one location, along with an ephemeris, to determine the independent station position referenced to the earth centered satellite coordinate system. The ephemeris is used may be either precise or broadcast and the coordinate reference frame and accuracy achieved are dependent on this choice. The use of precise ephemeris will give positions in the NSWC-9z2 reference system while broadcast ephemeris will give station positions in the NWL-1OD reference system. In this technique the position of the satellite in its orbit is assumed to be error-free and is held fixed in determining the position of the Doppler receiver.

Data from 35 to 40 passes, which can be collected in less than 4 days, are required for point positioning to yields results of better than 2m in each coordinate using broadcast ephemeris. While the precise ephemeris produces accuracy better than 1 meter.

### 9-9-2 Translocation Technique

Two Doppler receivers are operated simultaneously within a few hundred kilometers of each other. One of the occupied stations is of known coordinates. Satellite passes observed by both Doppler receiver are used in the computation of the relative positions of these stations with respect to each other. The basis for a translocation is that ephemeris errors will affect the positions of both stations in the same manner and, therefore, the relative accuracy between the two stations will be improved. The accuracy achieved by translocation varies with the inter-station distance between translocated stations. In case of precise ephemeris, the accuracy is about 0.3 to 0.5 meters plus 2 ppm of the inter-station distance. This is practically favorable for distances less than 500 km.

### 9-9-3 Short arc Technique

It requires the use of six or more receivers operating over the same period. Each pass of satellite must be observed by at least four of the receivers. A minimum of three station must be known in the short arc adjustment in addition to some orientation constraints. The accuracy of this method is about 0.5 to 0.25 meters.